Hypotraceable Digraphs

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ABSTRACT

A hypotraceable digraph is a digraph $D=(V, E)$ which is not traceable, i.e., does not contain a (directed)Hamiltonian path, but for which $D-v$ is traceable for all $v \in V$. We prove that a hypotraceable digraph of order n exists iff $n \geq 7$ and that for each $k \geq 3$ there are infinitely many hypotraceable oriented graphs with a source and a sink and precisely k strong components. We also show that there are strongly connected hypotraceable oriented graphs and that there are hypotraceable digraphs with precisely two strong components one of which is a source or a sink. Finally, we prove that hypo-Hamiltonian and hypotraceable digraphs may contain large complete subdigraphs.

1. INTRODUCTION

It was shown recently that hypo-Hamiltonian and hypotraceable graphs play an interesting role in combinatorial optimization since many of them induce facets of the monotone symmetric travelling salesman polytope, cf. [5]; furthermore, it has turned out that these graphs provide polyhedral reasons for the difficulty of the symmetric travelling salesman problem.

Hypo-Hamiltonian graphs have been studied in several papers (for references, see, e.g., [12]) and it has been shown among other things that such graphs may contain triangles [10] and large complete bipartite subgraphs [2] and that they may be planar and cubic [13]. The possible

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order of a cubic hypo-Hamiltonian graph has been completely determined [13], and it is left open for $n = 17$ whether or not there exists a hypo-Hamiltonian graph of order n [3].

Hypotraceable undirected graphs have been studied in [9, 11, 13]. The smallest known hypotraceable graph has order 34 [9], and the smallest known cubic hypotraceable is that of Horton (see [8]) which has order 40. In [13] an infinite family of cubic planar hypotraceable graphs is described.

Hypo-Hamiltonian (resp. hypotraceable) digraphs can be obtained from hypo-Hamiltonian (resp. hypotraceable) undirected graphs by replacing each edge by a 2-cycle except possible one which is replaced by a single directed edge. But there are simpler methods for constructing such graphs. It was proved independently by Thomassen [12], Fouquet and Jolivet [4], and Grötschel and Wakabayashi [6] that there is a hypo-Hamiltonian digraph or order n iff $n \geq 6$. Thomassen [12] also described an infinite family of hypo-Hamiltonian oriented graphs (i.e., digraphs with no 2-cycle).

In the present note we determine completely the possible order of a hypotraceable digraph and show that for each $k \geq 3$ there are infinitely many hypotraceable oriented graphs with a source and a sink and precisely k strong components. We also prove that there are strongly connected hypotraceable oriented graphs and that there are hypotraceable digraphs with precisely two strong components one of which is a source. Finally, we show that hypo-Hamiltonian or hypotraceable digraphs may contain large complete subdigraphs. The analogous problem for undirected graphs is still unsolved (see Chvátal [1]).

In a subsequent paper [7] the relationship of hypotraceable digraphs to the asymmetric travelling salesman problem will be studied. In particular, it will be shown that some of these digraphs indeed induce facets of the monotone asymmetric travelling salesman polytope.

2. CONSTRUCTIONS OF HYPOTRACEABLE DIGRAPHS

Theorem 1. There exists a hypotraceable digraph of order n iff $n \geq 7$. Furthermore, for each $k \geq 1$ there exist infinitely many hypotraceable digraphs with precisely k strong components.

Proof. It is easy to see that a hypotraceable digraph cannot have a vertex of outdegree (or indegree) 1. Also, if a digraph of order n contains a path with $n-1$ vertices and the remaining vertex has degree at least $n-1$, then the digraph has a Hamiltonian path. From these observations it follows immediately that a hypotraceable digraph cannot have order less than 6, and one can further prove that it cannot have order 6 either.

Now let n be a natural number, $n \geq 7$. Consider a hypo-Hamiltonian digraph D of order $n-1$ (by the afore-mentioned results such a digraph exists) and let v be any vertex of D. If we split v into a source v' and a sink v'' such that v' dominates the same vertices as v and v' is dominated by the same vertices as v, then it is easy to verify that the resulting digraph is hypotraceable and of order n.

This construction can be generalized as follows. Let $D_1, D_2, \ldots, D_{k-2}$ be pairwise disjoint hypo-Hamiltonian digraphs and let v_i be a vertex of D_i. Put $D_i' = D_i - v_i$ for $i = 1, 2, \ldots, k-2$. Let A_i (resp. B_i) be the set of vertices of D_i dominated by (resp. dominating) v_i. Now form the union $D_1' \cup D_2' \cup \cdots \cup D_{k-2}'$ and add two new vertices v', v''. Then we add

all edges from v'	to A_1,
all edges from $V(D_1')\backslash B_1$	to A_2,
all edges from B_1	to $V(D_2')\backslash A_2$,
all edges from $V(D_2')\backslash B_2$	to A_3,
etc., finally we add	
all edges from B_{k-2}	to v''.

Then it is apparent that the resulting digraph is hypotraceable, and it clearly has k strong components.

If D_1 and D_2 are hypotraceable digraphs with sources v_1', v_2', respectively, and sinks v_1'', v_2'', respectively, then we form the union $D_1 \cup D_2$ and we identify v_1' and v_2'' into a vertex and we identify v_2' and v_1'' into a vertex. It is easy to verify that the resulting digraph is strong and hypotraceable.

So it only remains to be proved that a hypotraceable digraph may have precisely two strong components. This follows from properties of so-called 2-fragments of hypotraceable undirected graphs. A 2-fragment of a hypotraceable graph H is an induced subgraph G of H containing two vertices x and y such that the following hold:

(i) G has no Hamiltonian path starting at x or y, and
(ii) for each vertex z of G, $G - z$ has a Hamiltonian path starting at x or y.

The existence of an infinite family of such graphs G follows from the existence of hypotraceable graphs of connectivity 2 (see [11, Lemma 5.1]). For every 2-fragment G of a hypotraceable graph considered in [11] one can also easily prove that

(iii) G has a Hamiltonian path.

Now consider a graph G satisfying (i), (ii), and (iii). Form the symmetric digraph associated with this graph and add a vertex v and the two edges from v to x and y. Then it is easy to see that the resulting digraph is hypotraceable and has a source. This completes the proof of the theorem. ∎

If the hypo-Hamiltonian digraphs used in the proof are all oriented graphs, then also the resulting hypotraceable digraphs are oriented graphs (except in the case $k=2$). Also, we can add all edges from D_i' to D_j' (in the case $k \geq 5$ in the proof above) whenever $i<j-1$ and the resulting digraph is still hypotraceable. So hypotraceable oriented graphs may contain large transitive tournaments.

Theorem 2. For each natural number k there exists a hypo-Hamiltonian (resp. hypotraceable) digraph containing a complete subdigraph of order k.

Proof. From the proof of Theorem 1 it follows that it is sufficient to describe hypo-Hamiltonian digraphs containing large complete subdigraphs.

Consider the Cartesian product of a cycle of length 2 and a cycle of odd length m. This digraph can be obtained from two disjoint cycles $x_1x_2 \cdots x_mx_1$ and $y_1y_2 \cdots y_my_1$ by adding all 2-cycles $x_iy_ix_i$. It is easy to prove that this digraph is hypo-Hamiltonian. Let A be any subset of $\{y_1, y_2, \ldots, y_m\}$ containing no two consecutive vertices. If we add all edges joining vertices of A, then the resulting digraph D is hypo-Hamiltonian. In order to see this, it is sufficient to prove that D has no Hamiltonian cycle. So assume that C is such a cycle and assume C contains an edge $x_{i-1}x_i$. If C does not contain the edge x_ix_{i+1}, then C contains the path $y_{i+1}x_{i+1}x_{i+2}$ and the edge x_iy_i. Since not both of y_i and y_{i+1} are in A, the edge y_iy_{i+1} must be in C. So we have proved that if C contains the edge $x_{i-1}x_i$ then either C contains the edge x_ix_{i+1} or the path $x_{i-1}x_iy_iy_{i+1}x_{i+1}x_{i+2}$. This leads to a contradiction and the proof is complete. ∎

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