DFG Forschungszentrum
Mathematik für Schlüsseltechnologien

Projekt B10

Describing polyhedra by few polynomial inequalities

Final Report

1 Members and duration of the project, cooperations

The project B10 "Describing polyhedra by few polynomial inequalities" took place from June 2003 to August 2004 under supervision of Prof. Martin Grötschel. In this time Hartwig Bosse held a position financed by the DFG at the TU-Berlin. Prof. Martin Henk, Universität Magdeburg, and Prof. Claus Scheiderer, Universität Konstanz, where involved in the project as external partners.

2 Target

A deep result in semi-algebraic geometry by Bröcker and Scheiderer ([Brö91], [Sch89]), states that any closed semi-algebraic set of dimension n can be described by at most $n(n+1)/2$ polynomial inequalities. Surprisingly, the number of polynomials necessary is completely independent of both the geometric and algebraic complexity of the corresponding set. All known proofs for this are non-constructive.

A class of semi-algebraic sets crucial to optimization are polyhedra. The main interest of the project was to find a constructive proof for the theorem of Bröcker and Scheiderer in the case of polyhedra. During the project, an appropriate construction was devised, yielding the description of a given polyhedron in $\mathbb{R}^n$ by $2n$ polynomial inequalities.

This new representation for polyhedra possibly offers new roads of attack for combinatorial optimization problems, involving techniques from nonlinear optimization. The corresponding approach could not be fully followed due to the short duration of the project and is therefore open to further research.

3 Background, results, outlook

3.1 Background

The power of linear programming, one of today's most important optimization techniques, is - to a large extent - based on deep insights into the interplay between the geometry and the algebraic description of polyhedra.
A linear program can be interpreted as the task to optimize a linear objective function restricted to a polyhedron. Consequently, all known algorithms solving linear programs incorporate some geometric properties of polyhedra, and their successful implementations use special analytic or algebraic representations of polyhedra: The simplex-algorithm is based on an efficient matrix-representation of vertices, inner-point-methods use analytic centers or the central path in polyhedra.

A special challenge in combinatorial optimization is the complexity of linear programs, arising from combinatorial problems. Often, the number of facets of the involved polyhedra is exponential in their dimension, and with this the number of linear inequalities used to describe them. There are techniques that deal with this efficiently – the ellipsoid-method in theory and in praxis the simplex-algorithm together with cutting plane techniques. Nevertheless, there is still potential for improvement here, subsequently leading to an interest in fundamentally new procedures.
In principle, the result of Bröcker and Scheiderer offers the possibility to describe polyhedra with exponentially many facets by quadratically many polynomial inequalities. Thus combinatorial optimization problems could be reformulated as the task to maximize a linear objective function over a system of *few* polynomial inequalities. The use of methods of nonlinear optimization could then lead to efficient numeric treatment of the resulting polynomials.

At the beginning of the project there were only few approaches for the construction of the appropriate polynomials: In his thesis [Ber98] BERNIG presented a description of 2-dimensional polygons using 2 polynomials, and in [GH03] GRÖTSCHEL and HENK present a constructive description of simple, convex polyhedra of arbitrary dimension n, where $O(n^n)$ many polynomial inequalities are needed. The Masters thesis of BOSSE [Bos03] however, suggested a generic, constructive way of describing polyhedra with $2n$ polynomial inequalities.

3.2 Results

The main target of the project B10, the constructive description of polyhedra by few polynomial inequalities was achieved. Using fundamental techniques, an algorithm was formulated, which supplies the description of any n dimensional polyhedron using $2n$ polynomial inequalities. This is close to the lower bound n for pointed polyhedra in $\mathbb{R}^n$. Thus additionally the maximum number of polynomials necessary to describe n dimensional polyhedra was reduced from quadratic in n to linear. The principles behind this algorithm are presented in [BGH04], the algorithm itself is presented in [Bos05].

3.3 Outlook

The obtained results offer the possibility of new approaches to apply methods from nonlinear optimization in combinatorial optimization:

The polynomials constructed in the algorithm obtained in Project B10 are based on sums of squares. For these there exists both a rich theoretical background (cf. [ML], [Las97]) and applicable algorithms (cf. [Las01]).

Unfortunately, a general disadvantage of the description of polyhedra using *few* polynomials is the high degree of the corresponding polynomials. In such a description, only the degree of the polynomials can reflect the geometric complexity of the polyhedron: If an n dimensional polyhedron with m facets is described by $2n$ polynomial inequalities, then at least one of the used polynomials has a degree of $m/(2n)$ or higher (cf. [GH03]). Since the number of facets m is independent of the dimension n, the degree of the resulting polynomials can obstruct their numeric treatment. However there is hope, to obtain a balance between the number and the degree of the used polynomials in the future. Accordingly, a goal of further research would be a description of polyhedra, where – *relatively to the number of facets* – few polynomials are needed.

The PhD thesis of Bosse [Bos05] suggests an additional approach for the description of special, n dimensional polytopes by means of n polynomial inequalities. In the considered case, the total degree of the used polynomials is bounded from above by the dimension and the number of facets of the polyhedron. A generalization of this would improve the numeric application of the resulting description. Mr. Bosse will continue working on both topics in a post-doc-position at the CWI in Amsterdam.

4 Publications and talks

4.1 Publications

The Representation of Polyhedra by Polynomial Inequalities
Martin Grötschel and Matrin Henk,
Discrete & Computational Geometry, 29:4 (2003) 485-504

Polynomial inequalities representing polyhedra
Hartwig Bosse, Martin Grötschel, and Matrin Henk,
Mathematical Programming, online, 2004

Representing polyhedra by few polynomial inequalities
Hartwig Bosse, PhD thesis
Handed in February 2005 at the TU-Berlin.

4.2 Talks

17. Nov, 2003: Hartwig Bosse *Describing Polyhedra By Polynomial Inequalities*
European Graduate Program „Combinatorics, Geometry, and Computation“, Berlin.

25. Nov, 2003: Hartwig Bosse *Semi-algebraische Beschreibung von Polyedern*
Post-graduate seminar „algebraische Geometrie“, Universität Duisburg.

12. Dez, 2003: Hartwig Bosse *Darstellung von Polyedern durch Polynomungleichungen*
Sächsischer Geometrietag 2003, Magdeburg.

13. Jan, 2004: Martin Grötschel, *Lineare Optimierung, Polyeder und semi-algebraische Geometrie*
Joint colloquium of the TU and the LMU München.

15. Apr, 2004: Hartwig Bosse, *Semi-algebraic representations of polyhedra*
Workshop „Algorithmic, Combinatorial and Applicable Real Algebraic Geometry“, MSRI Berkeley.

24. Jun, 2004: Hartwig Bosse, *Representing polyhedra by sums of squares*
Workshop „High Performance Optimization Techniques“, CWI Amsterdam.

23. Jul, 2004: Martin Grötschel, *What is going on in linear and integer programming?*
IX Encuentro de Matemáticas y sus Aplicaciones, Quito, Ecuador.

Literatur

[Ber98] A. Bernig, *Constructions for the Theorem of Bröcker and Scheiderer*, Master's thesis, Universität Dortmund, 1998.

[BGH04] Hartwig Bosse, Martin Grötschel, and Matrin Henk, *Polynomial inequalities representing polyhedra*, Mathematical Programming, published online s10107-004-0563-2 (2004).

[Bos03] H. Bosse, *Describing polyhedra by polynomial inequalities*, Master's thesis, Technische Universität Berlin, 2003.

[Bos05] ———, *Representing polyhedra by few polynomial inequalities*, Ph.D. thesis, Technische Universität Berlin, handed in (Feb 2005).

[Brö91] L. Bröcker, *On basic semi-algebraic sets*, Expo Math. **9** (1991), 289–334.

[GH03] Martin Grötschel and Matrin Henk, *The representation of polyhedra by polynomial inequalities*, Descrete and Computational Geometry **29** (2003), no. 4, 485–504.

[Las97] J.P. Lasserre, *Semidefinite programming vs. lp relaxations for polynomial programming*, Mathematics of Operations Research **27** (1997), 347–366.

[Las01] ———, *Global optimization with polynomials and the problems of moments*, SIAM Journal on Optimization **11** (2001), 798–817.

[ML] Franz Rendl Monique Laurent, *Semidefinite programming and integer programming*, To appear in the Handbook on Discrete Optimization edited by K.Aardal, G. Nemhauser and R. Weismantel.

[Sch89] C. Scheiderer, *Stability index of real varieties*, Inventiones Math **97** (1989), no. 3, 467–483.