Cuspidal plane curves of degree 12 and their Alexander polynomials

Diplomarbeit

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Remarks on the notation

The natural numbers $\mathbb{N}$ do not contain 0. The set $\mathbb{N} \cup \{0\}$ is denoted by $\mathbb{N}_0$. If a subset I of some ring S is an ideal of S, this will be indicated by $I \trianglelefteq S$. If S is a graded ring, then S_d denotes the set of homogeneous elements of degree $d \in \mathbb{Z}$. The same notation will be used for graded modules. For $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, set $\binom{n}{k} := 0$ whenever $n < k$.

Acknowledgements

I want to thank Prof. Kloosterman for introducing me to this interesting piece of mathematics. He has also always been available to answer my questions during the process of writing this thesis.

Außerdem möchte ich meinen Eltern danken, die mich während meines ganzen Studiums unterstützt haben. Schließlich geht mein Dank auch an Isabel, die in letzter Zeit viel Geduld mit mir aufbringen musste und trotzdem bei dieser Arbeit Korrektur gelesen hat.

All drawings in this thesis were created with Wolfram Mathematica ([29]).

1 Motivation

Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a curve in the complex projective plane. Then $\mathbb{P}^2_{\mathbb{C}} \setminus C$ is not only a Zariski-open subset of $\mathbb{P}^2_{\mathbb{C}}$, but also open with respect to the Euclidean topology. Moreover, $\mathbb{P}^2_{\mathbb{C}} \setminus C$ is path-connected. Thus one might ask for the fundamental group $\pi_1(\mathbb{P}^2_{\mathbb{C}} \setminus C)$.

Theorem 1.1 (Deligne-Fulton-Zariski)**.** *Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a reduced curve of degree $d \in \mathbb{N}$ with $r \in \mathbb{N}$ irreducible components. Assume further that the only singularities of C are ordinary double points. Then*

$$\pi_1(\mathbb{P}^2_{\mathbb{C}} \setminus C) \cong \mathbb{Z}^r / \langle (d_1, \ldots, d_r) \rangle,$$

where $d_i \in \mathbb{N}$ denotes the degree of the i-th irreducible component, $i = 1, \ldots, r$.

Zariski gave a proof of this theorem in 1929 (see [30, Theorem 9]). However, he relied on a statement of Severi, whose proof was incorrect. Severi's result was proven by Harris ([18]) in 1986. In the meantime, Deligne ([8, Théorème 1]) used methods of Fulton ([13]) to give an independent proof of the above theorem in 1979.

Thus the fundamental group of a curve complement seems to be related to the singularities of the curve. After ordinary double points, the next step is to allow ordinary cusps as singularities. In the sequel, such curves will be called cuspidal (plane) curves. Now the situation has become more sophisticated:

- The fundamental group is not necessarily abelian: For example, the complement of the three-cuspidal quartic given by

  $$C := V\left(x^2y^2 + x^2z^2 + y^2z^2 - 2xyz(x+y+z)\right) \subset \mathbb{P}^2_{\mathbb{C}}.$$

  has the fundamental group

  $$\pi_1(\mathbb{P}^2_{\mathbb{C}} \setminus C) \cong \langle a, b \mid a^2 = b^3 = (ab)^2 \rangle$$

  (see e. g. [9, Proposition 4.4.8] or [30, Section 9]). Nevertheless, if C is an irreducible curve of degree at most four and not a three-cuspidal quartic, then $\pi_1(\mathbb{P}^2_{\mathbb{C}} \setminus C)$ is an abelian group ([9, Exercise 4.4.1 and Proposition 4.4.3]).

- Consider the two curves

  $$\begin{aligned} C_1 &:= V\left((x^2 - 2y^2)^3 + (x^3 + z^3)^2\right), \\ C_2 &:= V\left((y^2 + \xi x^2)^3 - (x^2 + \xi y^2)^3 - (z^2 + \xi y^2 + \xi^2 x^2)^3\right), \end{aligned}$$

  where ξ is a primitive third root of unity. Both C_1 and C_2 are irreducible sextics with exactly six cusps as singularities. That is, the "combinatorial types" of C_1 and

C_2 coincide. However, the fundamental groups of the respective complements are non-isomorphic:

$$\pi_1(\mathbb{P}^2_{\mathbb{C}} \setminus C_1) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} \quad \ncong \quad \mathbb{Z}/6\mathbb{Z} \cong \pi_1(\mathbb{P}^2_{\mathbb{C}} \setminus C_2).$$

The reason is that all six cusps of C_1 lie on a conic, whereas the cusps of C_2 do not. This has already been discovered by Zariski, who also computed the two fundamental groups ([30, Section 9], [32, Section 9]). In general, if C_1 and C_2 are two plane curves with equal combinatorial data, but different fundamental groups, then (C_1, C_2) is called a Zariski pair.

As a consequence, the fundamental group of a curve does also depend on the position of its singularities. An invariant to decide whether two different curves yield different fundamental groups is given by a polynomial in $\mathbb{Q}[t]$, the so-called Alexander polynomial. In fact, the curves C_1 and C_2 mentioned in the above example lead to distinct polynomials.

However, the usefulness of Alexander polynomials is limited: If the degree of a cuspidal plane curve is not divisible by six, then its Alexander polynomial will always be trivial. Therefore, the first case where Alexander polynomials come into play is the case of sextics: If C is a cuspidal sextic, then its Alexander polynomial can be determined just by knowing the number κ of cusps of C, unless $\kappa = 6$. In the latter case, there are two possible polynomials depending on whether the six cusps lie on a conic or not.

Consequently, Alexander polynomials of cuspidal plane sextics are well understood. In the degree 12 case, it is much more difficult to predict the possible Alexander polynomials from the number of cusps: The currently known bounds are much coarser than in the case of sextics and cannot be expected to be sharp. The aim of this diploma thesis is to contribute various examples of cuspidal plane curves of degree 12 and to calculate their Alexander polynomials.

2 Cuspidal plane curves

2.1 Projective plane curves

This section collects some essential facts about curves in the complex projective plane. References for the mentioned results are e. g. [2] and [14].

Basic definitions. Consider the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$. Let $f = f(x, y, z) \in \mathbb{C}[x, y, z]$ be a square-free non-constant homogeneous polynomial. Then

$$V(f) := \{(\alpha : \beta : \gamma) \in \mathbb{P}^2_{\mathbb{C}} \mid f(\alpha, \beta, \gamma) = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$$

is called the (reduced) complex projective plane curve defined by f. This notion is well-defined. Sometimes, $f = 0$ will be called an equation for $V(f)$. A complex projective plane curve has an essentially unique square-free equation in the sense that if $V(f) = V(g)$ for square-free polynomials f and g, then there exists some unit $u \in \mathbb{C}^*$ such that $g = u \cdot f$. The degree of a complex projective plane curve C is defined to be the degree of any defining square-free polynomial and denoted by $\deg C$. If this polynomial can be chosen to be irreducible, then the corresponding curve is also called irreducible.

Multiplicity of a point and singularities. Let $C = V(f)$ be a complex projective plane curve of degree d and $p = (\alpha : \beta : \gamma)$ be a point in $\mathbb{P}^2_{\mathbb{C}}$. After applying a projective change of coordinates if necessary, assume that $\gamma = 1$. Write

$$f(x + \alpha, y + \beta, 1) = f_0(x, y) + \cdots + f_d(x, y), \quad \text{where } f_i \in \mathbb{C}[x, y]_i, \quad i = 1, \ldots, d.$$

The multiplicity of C at p is given by $m_p(C) := \min\{i \in \{0, \ldots, d\} : f_i \neq 0\}$. The point p lies on C if and only if $m_p(C) \geq 1$. The curve C is called smooth at p if $m_p(C) = 1$. A singular point of C is a point $p \in C$ with $m_p(C) \geq 2$. This happens if and only if the partial derivatives of f with respect to x, y and z vanish at p. If C contains no singular points, C is said to be smooth, otherwise C is called a singular curve.

Example: Let $p > q \geq 2$ be natural numbers. Set $f := x^p - y^q z^{p-q}$ and $C := V(f)$. Then $(0 : 0 : 1)$ is a singular point of C with multiplicity q.

Denote by $\Sigma \subset \mathbb{P}^2_{\mathbb{C}}$ the set of all singular points on C. Since Σ is cut out by f and its partial derivatives, it is a closed set in the Zariski topology. Moreover

$$2\#\Sigma \leq \sum_{p \in \Sigma} m_p(C)(m_p(C) - 1) \leq \begin{cases} d(d-1) & \text{and even} \\ (d-1)(d-2) & \text{if } f \text{ is irreducible.} \end{cases}$$

Intersection multiplicity. Let C and C' be two complex projective plane curves and $p := (0 : 0 : 1) \in \mathbb{P}^2_{\mathbb{C}}$. If $f(x, y, z)$ and $g(x, y, z)$ are the defining polynomials for C and C', respectively, then the intersection multiplicity of C and C' at p is defined via

$$i_p(C, C') := \dim_{\mathbb{C}} \mathbb{C}[x, y]_{\langle x,y\rangle}/(f(x, y, 1), g(x, y, 1)),$$

where $\mathbb{C}[x, y]_{\langle x,y\rangle}$ denotes the localization of $\mathbb{C}[x, y]$ at the prime ideal $\langle x, y\rangle$.

The intersection multiplicity satisfies $i_p(C, C') = 0$ if and only if $p \notin C \cap C'$. If p lies on a common component of C and C', then $i_p(C, C') = \infty$, otherwise $i_p(C, C')$ is finite. A lower bound for $i_p(C, C')$ is given by $m_p(C) \cdot m_p(C')$ for all $p \in C \cap C'$. Equality holds if and only if C and C' have no common tangent at p. Moreover, the intersection multiplicity does not depend on coordinate changes, thus it may be extended to arbitrary points in $\mathbb{P}^2_{\mathbb{C}}$.

A very useful statement on the intersection multiplicity is

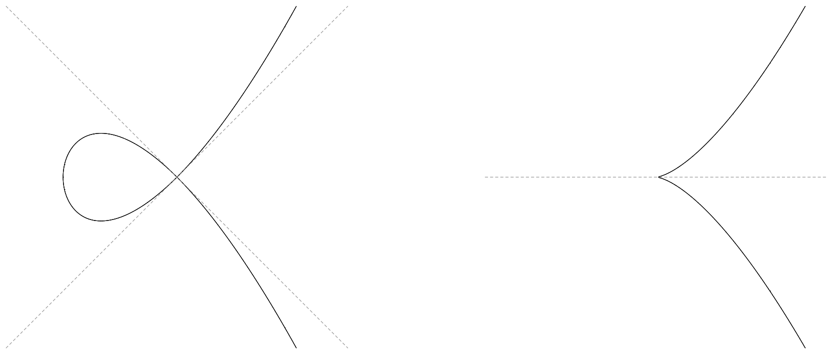
Theorem 2.1 (Bézout)**.** *Let C and C' be two complex projective plane curves without a common component. Then*

$$\sum_{p \in C \cap C'} i_p(C, C') = \deg C \cdot \deg C'.$$

Tangents, nodes and cusps. A line is a complex projective plane curve of degree 1. If $C = V(f)$ is a complex projective plane curve, then a line ℓ is called a tangent to C at $p \in \mathbb{P}^2_{\mathbb{C}}$ if $i_p(C, \ell) > m_p(C)$. A point $p \in C$ has $m_p(C)$ tangents, counted with multiplicity. These correspond to the linear factors of the homogeneous degree $m_p(C)$ part of $f(x + \alpha, y + \beta, 1)$ if $p = (\alpha : \beta : 1)$. Therefore, if p is a smooth point of C, then there is a unique tangent to C at p.

Now if $m_p(C) = 2$, than p has either two distinct tangents or a doubly counted tangent. In the first case, p is called an ordinary double point or ordinary node of C. In the latter case, p is called a cusp of C and the tangent line is called the cuspidal tangent. If additionally C intersects the cuspidal tangent ℓ with multiplicity $i_p(C, \ell) = 3$, then p is called an ordinary cusp.

Definition 2.2. *A complex projective plane curve C is called a cuspidal plane curve or simply a cuspidal curve if its singular points are either ordinary nodes or ordinary cusps.*



Ordinary node and ordinary cusp

Lemma 2.3 (Characterization of ordinary cusps)**.** *Let C be the complex projective plane curve defined by the polynomial $f(x,y,z) \in \mathbb{C}[x,y,z]_d$. Write*

$$f(x,y,1) = f_0(x,y) + \cdots + f_d(x,y), \quad \textit{where } f_i \in \mathbb{C}[x,y]_i, \quad i = 1,\ldots,d.$$

Then $(0:0:1)$ is a cusp if and only if

(a) $f_0(x,y) = f_1(x,y) = 0$,

(b) $f_2(x,y) = (ax+by)^2$ *for some* $(a,b) \in \mathbb{C}^2 \setminus \{(0,0)\}$,

(c) $f_3(x,y) \neq 0$ *and* $ax+by$ *does not divide* $f_3(x,y)$.

Proof:

($\Leftarrow$) Clearly, $p := (0:0:1)$ is a point of multiplicity two with a single tangent given by $\ell := V(ax+by)$ by (a) and (b). In particular $i_p(C,\ell) \geq m_p(C) + 1 = 3$. Since

$$\langle f(x,y,1), ax+by \rangle = \langle f_3(x,y) + \cdots + f_d(x,y), ax+by \rangle$$

as ideals in $\mathbb{C}[x,y]_{\langle x,y \rangle}$, and the affine curves $V(f_3(x,y)+\cdots+f_d(x,y))$ and $V(ax+by)$ have no common tangent at $(0,0)$ due to (c), this shows $i_p(C,\ell) = 3$.

($\Rightarrow$) If $p := (0:0:1)$ is an ordinary cusp, then $m_p(C) = 2$, which implies (a). Suppose that $\ell := V(ax+by+cz)$ is the cuspidal tangent at p, where $a,b,c \in \mathbb{C}$. Hence $ax+by+cz$ vanishes at $p = (0:0:1)$, thus $c = 0$. Since $m_p(C) = 2$, not both a and b can be zero. As $f_2(x,y)$ is the product of the tangent lines at p, (b) follows. Due to $i_p(C,\ell) = 3$, also (c) holds. $\square$

2.2 Plücker formulas and bounds on the number of cusps

Let $C = V(f)$ be a complex projective plane curve. If p is a smooth point of C, then there exists a unique tangent at p. Since the set of all singular points of C is finite, this gives rise to a rational map

$$\varphi : C \dashrightarrow \mathbb{P}^2_{\mathbb{C}}, \quad p \mapsto (a:b:c), \quad \text{where } V(ax+by+cz) \text{ is the tangent at } p.$$

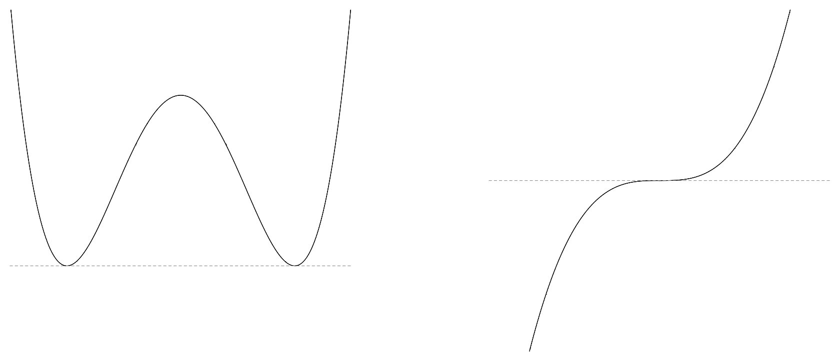
The closure of the image of φ is a complex projective plane curve and called the dual curve of C.

Proposition 2.4 (Plücker formulas)**.** *Let C be a cuspidal curve of degree $d \geq 2$ with κ cusps and δ nodes. Suppose that C has no lines as components. Assume further that the dual curve of C is also cuspidal with degree d^*, κ^* cusps and δ^* nodes. Then*

$$\begin{aligned} d^* &= d(d-1) - 2\delta - 3\kappa, \\ \kappa^* &= 3d(d-2) - 6\delta - 8\kappa, \\ \delta^* &= \frac{1}{2}d(d-2)(d^2-9) - (2\delta + 3\kappa)(d^2 - d - 6) + 2\delta(\delta - 1) + \frac{9}{2}\kappa(\kappa - 1) + 6\delta\kappa. \end{aligned}$$

Proof: Combine [2, Section 9.1, Theorem 1] and [2, Section 9.1, Remark (2) after Theorem 1]. □

Furthermore, the numbers δ^* and κ^* have another interpretation: A bitangent of C is a line that is tangent to C at more than one point. An inflection point or flex of C is a smooth point p of C, where the tangent line ℓ satisfies $i_p(C, \ell) = 3$. Now δ^* is the number of bitangents of C, whereas κ^* gives the number of inflection points.



Bitangent and inflectional tangent

Depending on the degree d, a curve is called line, conic, cubic, quartic, quintic, sextic, ... if $d = 1, 2, 3, 4, 5, 6, \ldots$, respectively. The Plücker formulas give rise to a bound on the number of cusps on a cuspidal curve:

Proposition 2.5 (First bound on the number of cusps)**.** *Let C be a cuspidal curve of degree $d \geq 2$ with κ cusps. Then C is either a three-cuspidal quartic, a nine-cuspidal sextic or*

$$\kappa \leq \frac{d(d-2)}{3}.$$

Proof: Suppose that C has δ nodes and that the dual curve is of degree d^* with κ^* cusps and δ^* nodes. The generalized Plücker formulas of [2, Section 9.1, Theorem 2] give the following relations:

$$\begin{aligned} d^* &= d(d-1) - 2\delta - 3\kappa, \\ \kappa^* &= 3d(d-2) - 6\delta - 8\kappa, \\ d &\leq d^*(d^*-1) - 2\delta^* - 3\kappa^*. \end{aligned}$$

Solving this with the constraints $d, d^*, \kappa, \kappa^*, \delta, \delta^* \in \mathbb{N}_0$, $d, d^* \geq 2$ and $\kappa > d(d-2)/3$ yields $(d, \kappa, \delta) = (4, 3, 0)$ or $(d, \kappa, \delta) = (6, 9, 0)$. Both three-cuspidal quartics and nine-cuspidal sextics exist, see Chapter 5. □

As a consequence, a complex projective plane curve C of degree d can have a cusp only if $d \geq 3$. If $d = 3$, then C has at most one cusp. An example is given by the cuspidal cubic $V(x^2z - y^3)$.

Moreover, a cuspidal curve of degree 12 can have at most 40 cusps. An equation for an irreducible curve of degree 12 with 39 cusps will be presented in Chapter 5 as Example (4.12).

If C is irreducible and has degree at least 15, the following improvement can be achieved (see [27, Formula (0.3)]):

Proposition 2.6 (Second bound on the number of cusps)**.** *Let C be an irreducible curve of degree $d \geq 6$ with κ cusps. Then*

$$\kappa \leq \frac{5}{16}d^2 - \frac{3}{8}d.$$

2.3 Analytic set germs

This section explains the connection between singular points of complex projective plane curves and analytic set germs following [2, III.8.2] and [15].

Analytic set germs. Let U be an open subset of $\mathbb{C}^n$ and $X \subseteq U$. X is called analytic at $x \in U$ if there are a neighborhood $V \subseteq U$ of x and holomorphic functions $f_1, \ldots, f_s$ on V such that

$$X \cap V = \{z \in V \mid f_1(z) = \cdots = f_s(z) = 0\}.$$

X is called an analytic subset of U if X is analytic at any $x \in U$.

Let U and U' be open subsets of $\mathbb{C}^n$ and $X \subseteq U$, $X' \subseteq U'$ be respective analytic subsets. Further let $x \in U \cap U'$. Then X and X' are said to be equivalent if there is an open neighborhood $V \subseteq U \cap U'$ of x such that $X \cap V = X' \cap V$. The corresponding equivalence class is called the analytic set germ at x and denoted by (X, x).

Mapping germs, analytic and topological equivalence. Let $(X, x) \subseteq (\mathbb{C}^m, x)$ and $(Y, y) \subseteq (\mathbb{C}^n, y)$ be analytic set germs, and let $U, V \subseteq \mathbb{C}^m$ be open neighborhoods of x. Suppose that $f : U \to \mathbb{C}^n$ and $g : V \to \mathbb{C}^n$ are two analytic mappings with $f(x) = g(x) = y$, which map representatives of X to representatives of Y. Then f and g define the same mapping germ $(X, x) \to (Y, y)$ if there is an open neighborhood $W \subseteq \mathbb{C}^m$ of x and a representative X' of X in W such that the restrictions of f and g to X' coincide.

Suppose $(X, 0)$ and $(Y, 0)$ are analytic set germs at $0 \in \mathbb{C}^n$ such that there is a mapping germ $\varphi : (\mathbb{C}^n, x) \to (\mathbb{C}^n, y)$ with $\varphi(X, x) = (Y, y)$. If φ is the germ of an analytic isomorphism, then (X, x) and (Y, y) are called analytically equivalent or sometimes contact equivalent. If

φ is the germ of a homeomorphism, then (X,x) and (Y,y) are called topologically equivalent. In particular, analytic equivalence of analytic set germs implies topological equivalence.

Application to complex projective plane curves. Suppose that $C \subset \mathbb{P}^2_{\mathbb{C}}$ is a complex projective plane curve with a point of multiplicity two at $p := (0:0:1)$. If C is defined by some homogeneous $f(x,y,z) \in \mathbb{C}[x,y,z]$, then consider the affine curve

$$C_z := V(f(x,y,1)) := \{(x,y) \in \mathbb{C}^2 \mid f(x,y,1) = 0\} \subset \mathbb{C}^2.$$

Now C_z is an analytic subset of $\mathbb{C}^2$ and $(C_z, (0,0))$ is an analytic set germ. If this set germ is analytically equivalent to $(V(x^2 - y^{k+1}), (0,0))$ for some $k \in \mathbb{N}$, then p is called an A_k singularity of C.

Lemma 2.7. *Ordinary nodes are A_1 singularities and ordinary cusps are A_2 singularities.*

Proof: Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a complex projective plane curve and $p := (0:0:1)$.

(a) If p is an ordinary double point, then by the discussion in Section 2.1,

$$f(x,y,1) = (ax+by)(cx+dy) + \text{higher order terms},$$

where $(a,b),(c,d) \in \mathbb{C}^2$ are linearly independent. Thus the Hessian matrix

$$\begin{pmatrix} \frac{\partial^2 f(x,y,1)}{\partial x^2}(0,0) & \frac{\partial^2 f(x,y,1)}{\partial x \partial y}(0,0) \\ \frac{\partial^2 f(x,y,1)}{\partial y \partial x}(0,0) & \frac{\partial^2 f(x,y,1)}{\partial y^2}(0,0) \end{pmatrix} = \begin{pmatrix} 2ac & ad+bc \\ ad+bc & 2bd \end{pmatrix}$$

has determinant $(ad-bc)^2 \neq 0$ and therefore full rank. By the Morse lemma [15, Theorem 2.46], this means that p is an A_1 singularity.

(b) If p is an ordinary cusp, then after applying a coordinate change,

$$f(x,y,1) = x^2 + f_3(x,y) + \text{higher order terms},$$

where x does not divide $f_3(x,y)$ by Lemma 2.3. The Hessian matrix

$$\begin{pmatrix} \frac{\partial^2 f(x,y,1)}{\partial x^2}(0,0) & \frac{\partial^2 f(x,y,1)}{\partial x \partial y}(0,0) \\ \frac{\partial^2 f(x,y,1)}{\partial y \partial x}(0,0) & \frac{\partial^2 f(x,y,1)}{\partial y^2}(0,0) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

has now rank one. Moreover,

$$f_x := \frac{\partial f(x,y,1)}{\partial x} = 2x + \frac{\partial f_3(x,y)}{\partial x} + \text{higher order terms},$$
$$f_y := \frac{\partial f(x,y,1)}{\partial y} = \frac{\partial f_3(x,y)}{\partial y} + \text{higher order terms}.$$

Now $(0,0)$ is a common point of the affine curves $V(f_x)$ and $V(f_y)$. Due to the divisibility condition $x \nmid f_3(x,y)$, both curves have distinct tangents at $(0,0)$. Since $(0,0)$ is a smooth point on $V(f_x)$ and a double point on $V(f_y)$, this implies

$$\dim_{\mathbb{C}} \mathbb{C}[x,y]_{\langle x,y\rangle} / \langle f_x, f_y \rangle = 1 \cdot 2 = 2.$$

By [15, Theorem 2.48], p is an A_2 singularity. $\square$

2.4 Kummer coverings

The aim of this section is to investigate the behavior of a curve and its singularities, when its defining polynomial $f(x, y, z) \in \mathbb{C}[x, y, z]$ is replaced by $f(x^n, y^n, z^n)$ for some natural number $n \in \mathbb{N}$.

The Kummer covering φ_n. For $n \in \mathbb{N}$ consider the morphism

$$\varphi_n : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}, \quad (x : y : z) \mapsto (x^n : y^n : z^n).$$

The map $\varphi_n : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ is a finite surjective morphism of degree n^2. In particular, the cardinality of each fiber is bounded from above by n^2. More precisely, if $p \in \mathbb{P}^2_{\mathbb{C}}$, then

$$\#\varphi_n^{-1}(p) = \begin{cases} 1 & \text{if } p \in \{(1:0:0), (0:1:0), (0:0:1)\}, \\ n & \text{if } p \in V(xyz) \setminus \{(1:0:0), (0:1:0), (0:0:1)\}, \\ n^2 & \text{otherwise.} \end{cases}$$

If p is one of the three points $(1 : 0 : 0)$, $(0 : 1 : 0)$ or $(0 : 0 : 1)$, then p has one preimage under φ_n, namely p. Otherwise, if p is none of these points, but still lies on $V(xyz)$, then p has n preimages: Indeed, suppose without loss of generality that $p = (0 : 1 : \alpha)$ for some $\alpha \in \mathbb{C}^*$. Then

$$\varphi_n^{-1}(p) = \{(0 : \beta : \gamma) \mid \beta^n = 1 \text{ and } \gamma^n = \alpha\}.$$

If such a pair (β, γ) is given and ξ is an n-th root of unity, then $(0 : \xi\beta : \xi\gamma)$ and $(0 : \beta : \gamma)$ define the same point in the complex projective plane. Consequently, $\varphi_n^{-1}(p)$ consists of n distinct points.
If p is of the form $(1 : \alpha : \beta)$ for some $\alpha, \beta \in \mathbb{C}^*$, a similar argument shows that $\#\varphi_n^{-1}(p) = n^2$.

Since the cardinality of the fibers drops at $V(xyz)$, this set is called the ramification locus of φ_n. Outside the ramification locus, i. e., when restricted to $\mathbb{P}^2_{\mathbb{C}} \setminus V(xyz)$ on both sides, φ_n is a covering map of degree n^2 with respect to the Euclidean topology. Furthermore, the corresponding field extension is a Kummer extension with Galois group $(\mathbb{Z}/n\mathbb{Z})^2$. For these reasons, φ_n is called a Kummer covering.

Pulling back curves via φ_n. Let $f \in \mathbb{C}[x, y, z]$ be a square-free homogeneous polynomial. Then the homogeneous ideal

$$J(f) := \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \trianglelefteq \mathbb{C}[x, y, z]$$

is called the Jacobian ideal of f. By the projective Nullstellensatz (see e. g. [19, Section I.2]) and Euler's theorem (see e. g. [2, II.4.4, Theorem 3]), the radical ideal of $J(f)$ is the ideal of singular points of the curve $V(f)$.

Note that the morphism φ_n induces a ring homomorphism

$$\varphi_n^* : \mathbb{C}[x, y, z] \to \mathbb{C}[x, y, z], \quad g(x, y, z) \mapsto g(x^n, y^n, z^n).$$

Proposition 2.8. *Let $C = V(f) \subset \mathbb{P}^2_{\mathbb{C}}$ be a complex projective plane curve of degree $d \in \mathbb{N}$. Suppose that C intersects $V(xyz)$ in smooth points only and that C does not contain any of the points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$.*

Then $\varphi_n^{-1}(C) = V(\varphi_n^*(f)) \subset \mathbb{P}^2_{\mathbb{C}}$ is a curve of degree nd and

$$J(\varphi_n^*(f)) = \left\langle x^{n-1} \cdot \varphi_n^* \left(\frac{\partial f}{\partial x}\right), y^{n-1} \cdot \varphi_n^* \left(\frac{\partial f}{\partial y}\right), z^{n-1} \cdot \varphi_n^* \left(\frac{\partial f}{\partial z}\right) \right\rangle.$$

In particular, for the singular loci $\Sigma(C)$ resp. $\Sigma(\varphi_n^{-1}(C))$ holds

$$\Sigma(\varphi_n^{-1}(C)) = \varphi_n^{-1}(\Sigma(C) \cup \Delta),$$

where Δ is the set of points where C intersects $V(xyz)$ with multiplicity at least two.

Proof: If C is defined by the vanishing of $f = f(x, y, z)$, then it is clear that $\varphi_n^{-1}(C)$ is a curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree nd with defining equation $\varphi_n^*(f) = f(x^n, y^n, z^n) = 0$. Note that

$$\frac{\partial}{\partial x} f(x^n, y^n, z^n) = nx^{n-1} \cdot \frac{\partial f}{\partial x}(x^n, y^n, z^n) = nx^{n-1} \cdot \varphi_n^* \left(\frac{\partial f}{\partial x}(x, y, z)\right).$$

This holds analogously for the derivatives with respect to y and z. In particular, the ideal $J(\varphi_n^*(f))$ is of the described form.

Now let $(\alpha : \beta : \gamma)$ be a singular point of $\varphi_n^{-1}(C)$. Then from the description of $J(\varphi_n^*(f))$ follows that either $\varphi_n(\alpha : \beta : \gamma)$ is a singular point of C or $(\alpha : \beta : \gamma)$ lies on $V(xyz)$. In the latter case, note at first that $(\alpha : \beta : \gamma)$ is by assumption none of the points $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$. Thus only one coordinate may vanish, assume without loss of generality that $\alpha = 0$. In other terms,

$$\frac{\partial f}{\partial x}(\alpha^n, \beta^n, \gamma^n) \neq 0, \quad \frac{\partial f}{\partial y}(\alpha^n, \beta^n, \gamma^n) = 0, \quad \frac{\partial f}{\partial z}(\alpha^n, \beta^n, \gamma^n) = 0.$$

This shows that the tangent to C at $(\alpha^n : \beta^n : \gamma^n) \in C \cap V(x)$ is given by $V(x)$. In particular, C intersects $V(x)$ at $(\alpha^n : \beta^n : \gamma^n)$ with multiplicity at least 2, hence $(\alpha : \beta : \gamma) \in \varphi_n^{-1}(\Delta)$.

If $p \in \Sigma(C)$ is a singular point of C, then the description of $J(\varphi_n^*(f))$ shows that any point in $\varphi_n^{-1}(p)$ is in $\Sigma(\varphi_n^{-1}(C))$. If $q \in \Delta$, say without loss of generality $q = (0 : 1 : \alpha)$ for some $\alpha \in \mathbb{C}^*$, then the tangent to C at q is given by $V(x)$. Since all points in $\varphi_n^{-1}(q)$ have a vanishing first coordinate, they are all singular points of $\varphi_n^{-1}(C)$. □

Singularities of $\varphi_n^{-1}(C)$ ([1, Section 5]). How do the singularities of $\varphi_n^{-1}(C)$ look like? The answer is given in the following two lemmas:

Lemma 2.9. *Let $p \in \Sigma(C)$ be a singular point of $C = V(f)$. Then the germs (C, p) and $(\varphi_n^{-1}(C), q)$ are analytically isomorphic for all n^2 points $q \in \varphi^{-1}(p)$.*

Proof: Note that φ_n is unramified in a neighborhood of $p = \varphi_n(q)$, as C has no singular points on the ramification locus $V(xyz)$. Let $q = (\alpha : \beta : 1)$ for some $\alpha, \beta \in \mathbb{C}^*$ and define $g(x, y, z) := f(x^n, y^n, z^n)$. Then $g(x+\alpha, y+\beta, 1)$ is a local equation for $\varphi_n^{-1}(C)$ around q. Applying the analytic coordinate changes

$$\begin{aligned} (x, y) &\mapsto (x', y') := (x-\alpha, y-\beta), \\ (x', y') &\mapsto (x'', y'') := (\sqrt[n]{x'}, \sqrt[n]{y'}), \\ (x'', y'') &\mapsto (x'' + \alpha^n, y'' + \beta^n) \end{aligned}$$

yields $f(x+\alpha^n, y+\beta^n, 1)$, which is a local equation for C around p. □

For singularities coming from the set Δ, one has:

Lemma 2.10. *Let $p \in \Delta$ be a point where C intersects $V(xyz)$ with multiplicity $e \geq 2$. Then all n singularities in $\varphi_n^{-1}(p)$ are topologically equivalent to $(V(x^n - y^e), 0)$.*

Proof: Take $q \in \varphi_n^{-1}(p)$ and suppose without loss of generality that $q = (0 : \alpha : 1)$ for some $\alpha \in \mathbb{C}^*$. Let $g(x, y, z) := f(x^n, y^n, z^n)$. Then $g(x, y+\alpha, 1)$ is a local equation for $\varphi_n^{-1}(C)$ around q. Applying the coordinate changes

$$\begin{aligned} (x, y) &\mapsto (x', y') := (x, y-\alpha), \\ (x', y') &\mapsto (x'', y'') := (x', \sqrt[n]{y'}), \\ (x'', y'') &\mapsto (x'', y'' + \alpha^n) \end{aligned}$$

gives $f(x^n, y+\alpha^n, 1)$. Finally note that $f(x, y+\alpha^n, 1)$ can be transformed into $x - y^e$, since C intersects $V(x)$ with multiplicity e at the smooth point $p = (0 : \alpha^n : 1)$. □

Observe that it is always possible to find a change of coordinates such that $\Delta = \emptyset$. For example, if C is a cuspidal curve of degree d with κ cusps and $\Delta = \emptyset$, then Proposition 2.8 and Lemma 2.9 state that $\varphi_n^{-1}(C)$ is a cuspidal curve of degree nd with κn^2 cusps. So φ_n enables to produce cuspidal curves with a high number of cusps. Using Lemma 2.10, if $\Delta \neq \emptyset$ and the intersection with $V(xyz)$ is sufficiently nice, then one can put even more cusps into $\varphi_n^{-1}(C)$. This will be exploited in Chapter 5.

2.5 Elliptic threefolds and Mordell-Weil rank

The results of this section are mainly taken from [21] and [22].

Elliptic threefolds and their Mordell-Weil group. An elliptic threefold is a quadruple (X, S, π, σ_0), where X is a smooth projective threefold, S a smooth projective surface, $\pi : X \to S$ a flat morphism such that the generic fiber is a curve of genus one and $\sigma_0 : S \to X$ is a section of π. The morphism π is also called the elliptic fibration of the elliptic threefold.

Suppose that S is a rational surface. In particular, its function field is given by $\mathbb{C}(x, y)$. Now the function field of X, $\mathbb{C}(X)$, becomes a field extension of $\mathbb{C}(x, y)$ via the morphism π. Moreover, $\mathbb{C}(X)$ is the function field of an elliptic curve E over $\mathbb{C}(x, y)$, i.e., $\mathbb{C}(X) = \mathbb{C}(u, v, x, y)$.

Choosing a globally minimal Weierstrass equation, there are polynomials $f, g \in \mathbb{C}[x, y]$ such that

$$E = V(-v^2 + u^3 + ug(x, y) + f(x, y)) \subset \mathbb{C}(x, y)^2$$

and no non-constant polynomial $h \in \mathbb{C}[x, y]$ satisfies $h^4 \mid g$ and $h^6 \mid f$.

The set of all rational sections $S \dashrightarrow X$ of π carries a group structure. This group is called the Mordell-Weil group of $\pi : X \to S$. It is isomorphic to the group of $\mathbb{C}(x, y)$-rational points on the elliptic curve E. The identity element is given by the section σ_0. The Mordell-Weil group is a birational invariant.

Elliptic threefolds from cuspidal curves. Let C be a cuspidal curve of degree $6k$, $k \in \mathbb{N}$, and pick a square-free equation $f \in \mathbb{C}[x, y, z]_{6k}$. Choose coordinates u, v, x, y, z on $\mathbb{P}^4_{\mathbb{C}}$ with weights $2k, 3k, 1, 1, 1$, respectively, and consider the hypersurface

$$W_f := V(-v^2 + u^3 + f(x, y, z)) \subset \mathbb{P}^4_{\mathbb{C}}(2k, 3k, 1, 1, 1).$$

Let $p : W_f \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ be the projection from $(1 : 1 : 0 : 0 : 0) \in W_f$ onto the plane $V(u, v)$. Pick now a point $q \in \mathbb{P}^2_{\mathbb{C}}$. If $q \notin C$, then $\overline{p^{-1}(q)}$ is an elliptic curve with j-invariant zero. Otherwise, $\overline{p^{-1}(q)}$ is a cuspidal cubic. Furthermore, W_f is birational to an elliptic threefold $X_f := (X, S, \pi, \sigma_0)$, where S is a rational surface and $\pi : X \to S$ is birational to p.

The Mordell-Weil group $\mathrm{MW}(X_f)$ of an elliptic threefold associated to a cuspidal curve $V(f)$ of degree $6k$ is always finitely generated. In [24, Theorem 3.11], the following bound for its rank is shown:

Lemma 2.11. *Under the above hypotheses,* $\mathrm{rk}_{\mathbb{Z}} \mathrm{MW}(X_f) \leq 10k - 2$.

This bound will be improved in Chapter 4. The rank of the Mordell-Weil group can be computed more explicitly:

Theorem 2.12. *Let $C = V(f)$ be a cuspidal curve of degree $6k$ for some $k \in \mathbb{N}$. Denote by Σ the locus of cusps of C. Then the following numbers are equal:*

(a) The $\mathbb{Z}$-rank of the Mordell-Weil group of the elliptic threefold X_f,

(b) $2 \dim_{\mathbb{C}} \operatorname{coker} \varphi$, where

$$\varphi : \mathbb{C}[x, y, z]_{5k-3} \to \mathbb{C}^{\#\Sigma}, \quad f \mapsto (f(p))_{p \in \Sigma},$$

(c) $2 \dim_{\mathbb{C}} \operatorname{coker} \psi$, where

$$\psi : \mathbb{C}[x, y, z]_{7k-3} \to \mathbb{C}^{2\#\Sigma}, \quad f \mapsto \left(f(p), \frac{\partial f}{\partial \ell_p}(p) \right)_{p \in \Sigma}$$

and $\ell_p = 0$ is a square-free equation for the cuspidal tangent at $p \in \Sigma$.

Idea of proof: Following [21, Proposition 3.1] and [22], there is a map

$$\theta : H^4(\mathbb{P}^4_{\mathbb{C}}(2k, 3k, 1, 1, 1) \setminus W_f) \to H^4_\Sigma(W_f).$$

such that the dimension of the cokernel of θ equals $\operatorname{rk}\mathrm{MW}(\pi)$. If ω is a primitive third root of unity, then

$$\vartheta_\omega : W_f \to W_f, \quad (u : v : x : y : z) \mapsto (\omega u : v : x : y : z)$$

is an automorphism of W_f leaving Σ pointwise fixed. Now θ is ϑ^*_ω-equivariant, so its cokernel may be decomposed into the eigenspaces of $1, \omega$ and ω^2. The 1-eigenspace is trivial and the eigenspaces of ω and ω^2 have the same dimension. The ω-eigenspace of the cokernel of θ has dimension (b), whereas the ω^2-eigenspace has dimension (c).

As a consequence, the Mordell-Weil rank of X_f is always an even number.

Quasi-toric relations ([24, Section 4]). Let $C = V(f)$ be a cuspidal curve of degree $6k$ and X_f an elliptic threefold birational to W_f. Then $\mathrm{MW}(X_f)$ is isomorphic to the group of $\mathbb{C}(x, y)$-rational points of the elliptic curve E defined by the polynomial

$$-v^2 + u^3 + f(x, y, 1) \in \mathbb{C}(x, y)[u, v].$$

Let (α, β) be a $\mathbb{C}(x, y)$-rational point of E. Then there exist polynomials $g_1, g_2, g_3 \in \mathbb{C}[x, y]$ such that $\alpha = g_1/g_3$ and $\beta = g_2/g_3$ and

$$\frac{g_2^2}{g_3^2} - \frac{g_1^3}{g_3^3} = f(x, y, 1).$$

Multiplying with g_3^6 yields that

$$(g_2 g_3^2)^2 + (-g_1 g_3)^3 = g_3^6 \cdot f(x, y, 1).$$

Homogenizing, one obtains that any $\mathbb{C}(x, y)$-rational point of E gives polynomials h_1, h_2, h_3 in $\mathbb{C}[x, y, z]$ satisfying

$$h_1^2 + h_2^3 = h_3^6 \cdot f.$$

Conversely, if such polynomials h_1, h_2, h_3 exist, this gives a $\mathbb{C}(x, y)$-rational point of E.

Proposition 2.13. *Let $f \in \mathbb{C}[x, y, z]_{6k}$ be irreducible and suppose that $V(f)$ is a cuspidal curve. Then the set*

$$\{(h_1, h_2, h_3) \in \mathbb{C}[x, y, z]^3 \mid h_1^2 + h_2^3 = h_3^6 \cdot f\}$$

is called the set of quasi-toric relations of type $(2, 3, 6)$ of f. Moreover, this set carries a group structure isomorphic to $\operatorname{rk}\mathrm{MW}(X_f)$ copies of $\mathbb{Z}$.

Proof: This is Theorem 4.7 in [24]. □

3 Alexander polynomials

This chapter is inspired by [23] and [9, Chapter 2]. It introduces the Alexander polynomial in general and for knots and curves in detail. The last section connects the Alexander polynomial of a cuspidal curve with the Mordell-Weil group of an associated elliptic threefold.

3.1 Definition

Let X be a topological space with the homotopy type of a finite CW complex. Suppose that there is a group epimorphism $\varepsilon : \pi_1(X) \twoheadrightarrow \mathbb{Z}$. Then $\ker \varepsilon$ gives rise to a covering $\tilde{X} \to X$, called the infinite cyclic covering, such that the group of deck transformations $\mathrm{Aut}(\tilde{X}/X)$ satisfies

$$\mathrm{Aut}(\tilde{X}/X) \cong \pi_1(X)/\ker \varepsilon \cong \mathbb{Z}.$$

Fix a generator T of $\mathrm{Aut}(\tilde{X}/X)$. Then the first homology group of $\tilde{X}$ with rational coefficients becomes a $\mathbb{Q}[t, t^{-1}]$-module via

$$\mathbb{Q}[t, t^{-1}] \times H_1(\tilde{X}, \mathbb{Q}) \to H_1(\tilde{X}, \mathbb{Q}), \quad t \cdot c \mapsto H_1(\tilde{X}, \mathbb{Q})(T)(c),$$

where $H_1(\tilde{X}, \mathbb{Q})(T)$ is the map obtained by the functoriality of H_1.

$\mathbb{Q}[t, t^{-1}]$ is the localization of the principal ideal domain $\mathbb{Q}[t]$ at the powers of t, hence itself a principal ideal domain. Furthermore, $H_1(\tilde{X}, \mathbb{Q})$ is finitely generated as a $\mathbb{Q}[t, t^{-1}]$-module, as it has the homotopy type of a finite CW complex. Thus by the classification of finitely generated modules over principal ideal domains, there are elements $d_1, \dots, d_k \in \mathbb{Q}[t, t^{-1}]$ with $d_1 \mid \cdots \mid d_k$ such that

$$H_1(\tilde{X}, \mathbb{Q}) = \mathbb{Q}[t, t^{-1}]^r \oplus \mathbb{Q}[t, t^{-1}]/\langle d_1 \rangle \oplus \cdots \oplus \mathbb{Q}[t, t^{-1}]/\langle d_k \rangle.$$

This motivates the following definition:

Definition 3.1. *The Alexander polynomial $\Delta_{X,\varepsilon} \in \mathbb{Q}[t]$ of X relative to the epimorphism $\varepsilon : \pi_1(X) \twoheadrightarrow \mathbb{Z}$ is defined as follows:*

(a) If $r = 0$, i. e., $H_1(\tilde{X}, \mathbb{Q})$ is a torsion module over $\mathbb{Q}[t, t^{-1}]$, then set

$$\Delta_{X,\varepsilon}(t) := d_1 \cdots d_k \cdot u \cdot t^m,$$

where $u \in \mathbb{C}^*$, $m \in \mathbb{Z}$ are so chosen that $\Delta_{X,\varepsilon} \in \mathbb{Q}[t]$, $\Delta_{X,\varepsilon}(0) \neq 0$ and $\Delta_{X,\varepsilon}(1) = 1$ if $\Delta_{X,\varepsilon}(1) \neq 0$.

(b) If $r > 0$, then define $\Delta_{X,\varepsilon} := 0$.

3.2 Knots and links

A link is a closed smooth submanifold in the three-dimensional sphere $\mathbb{S}^3$ such that each connected component is diffeomorphic to the circle. A knot is a connected link.

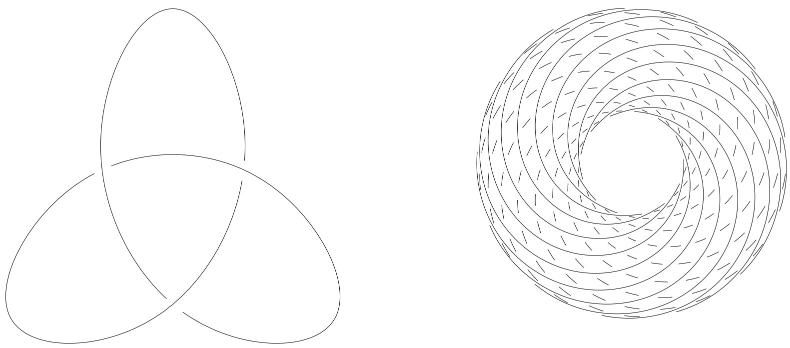
Example: Let p, q be coprime natural numbers. Consider the map

$$\psi : \mathbb{C} \to \mathbb{C}^2, \quad w \mapsto \left(\frac{w^q}{\sqrt{2}}, \frac{w^p}{\sqrt{2}} \right).$$

If $w \in \mathbb{S}^1$, i. e., $|w| = 1$, then

$$|\psi(w)|^2 = \left| \frac{w^q}{\sqrt{2}} \right|^2 + \left| \frac{w^p}{\sqrt{2}} \right|^2 = \frac{|w|^{2q}}{2} + \frac{|w|^{2p}}{2} = 1.$$

Hence $\psi(\mathbb{S}^1) \subseteq \mathbb{S}^3$. Moreover $\psi|_{\mathbb{S}^1}$ is an embedding of $\mathbb{S}^1$ into $\mathbb{S}^3$. Its image is called the (p, q)-torus knot. The $(2, 3)$-torus knot is also known as trefoil knot.



Trefoil knot and $(17, 13)$-torus knot

If L is a link with $m \in \mathbb{N}$ connected components. then $H_1(\mathbb{S}^3 \setminus L, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^m$ ([9, Chapter 2, Lemma 1.4]). In particular, if K is a knot and $X := \mathbb{S}^3 \setminus K$, then Hurewicz's theorem yields a group epimorphism

$$\varepsilon : \pi_1(X) \twoheadrightarrow H_1(X, \mathbb{Z}) \xrightarrow{\simeq} \mathbb{Z}.$$

Now, as shown in [23, Section 2], $H_1(\tilde{X}, \mathbb{Q})$ becomes a torsion module over $\mathbb{Q}[t, t^{-1}]$. Thus the corresponding Alexander polynomial $\Delta_K := \Delta_{X,\varepsilon}$ is nonzero. It is called the Alexander polynomial of the knot K.

Example: Let K be the (p, q)-torus knot for coprime numbers $p, q \in \mathbb{N}$. Using the van Kampen theorem, one can show (see [9, Chapter 2, Example 1.6])

$$\pi_1(\mathbb{S}^3 \setminus K) \cong \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z} \cong \langle a, b \mid a^p = b^q \rangle.$$

Using Fox calculus ([12]), the Alexander polynomial of K can be determined starting from a presentation for the fundamental group. In this case this procedure yields ([9, Example

2.1.13])

$$H_1(\tilde{X}, \mathbb{Q}) \cong \mathbb{Q}[t, t^{-1}]/(\Delta_{p,q}(t)), \quad \text{where } \Delta_{p,q}(t) := \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)} \in \mathbb{Q}[t].$$

In particular, for the trefoil knot holds $\Delta_{2,3}(t) = t^2 - t + 1$.

3.3 Alexander polynomials associated to curves

Links from plane curve singularities. Let $f \in \mathbb{C}[x, y, z]_d$ be a square-free homogeneous polynomial of degree $d \in \mathbb{N}$ and suppose that the complex projective plane curve $V(f)$ intersects $V(z)$ transversely, i. e. $V(f) \cap V(z)$ consists of d distinct points. Assume further that $V(f)$ is singular at $(0 : 0 : 1)$. Define the affine plane curve $C_z := V(f(x, y, 1)) \subset \mathbb{C}^2$. Now the 3-sphere $\mathbb{S}^3$ intersects C_z in a 1-sphere, which produces a link.

Example: Suppose $f := x^p - y^q z^{p-q}$ for coprime natural numbers $p > q$. Then C_z is defined by the vanishing of $x^p - y^q$. That is, C_z is the image of

$$\psi' : \mathbb{C} \to \mathbb{C}^2, \quad w \mapsto (w^q, w^p)\,.$$

We have seen the quite similar map ψ before in 3.2. Hence analogously to ψ, the embedding $\psi'|_{\mathbb{S}^1}$ yields a knot which is equivalent to the (p, q)-torus knot.

In general, any singular point of a complex projective plane curve is associated with a link. Moreover, two singularities are topologically equivalent if and only if their induced links are equivalent (see [2, Section III.8.3]).

Three Alexander polynomials. We will define three different Alexander polynomials associated to a curve following [23] and [26].

Definition 3.2. *Let C be a complex projective plane curve and p be a singular point of C. The Alexander polynomial of the induced link is called the local Alexander polynomial of C at p and denoted by $\Delta_{C,p}$.*

Lemma 3.3. *Let C be a cuspidal curve with κ cusps and δ nodes. Denote by Σ the set of all singular points of C. Then*

$$\prod_{p \in \Sigma} \Delta_{C,p}(t) = (t-1)^{\delta}(t^2 - t + 1)^{\kappa}.$$

Proof: Suppose that p is a cusp of C. Then the corresponding analytic set germ is analytically equivalent to $(V(x^2 - y^3), (0, 0))$ by Lemma 2.7. Since analytic equivalence implies topological equivalence, p induces a (2,3)-torus knot, i. e. a trefoil knot in virtue of the above example. Hence $\Delta_{C,p}(t) = t^2 - t + 1$.

If p is a node of C, then the corresponding analytic set germ is analytically equivalent to $(V(x^2 - y^2), (0, 0))$ also by Lemma 2.7. This set germ is not irreducible and the induced link is not a knot. However, this link is rather simple, as it is equivalent to two linked circles ([2, Proposition 13 in II.8.3]). The resulting Alexander polynomial is $t-1$ (see e. g. [9, Theorem 6.4.5]). □

Let C be a complex projective plane curve that intersects $V(z)$ transversely and consider the associated affine plane curve C_z. Note that

$$\pi_1(\mathbb{C}^2 \setminus C_z) \cong \pi_1(\mathbb{P}^2_{\mathbb{C}} \setminus (C \cup V(z))).$$

If r is the number of irreducible components of C_z, then the first singular homology group of $\mathbb{C}^2 \setminus C_z$ is isomorphic to $\mathbb{Z}^r$, as C_z is an affine curve ([9, Corollary 4.1.4]). Using Hurewicz's theorem, this motivates a group epimorphism

$$\varepsilon : \pi_1(\mathbb{C}^2 \setminus C_z) \twoheadrightarrow H^1(\mathbb{C}^2 \setminus C_z, \mathbb{Z}) \xrightarrow{\simeq} \mathbb{Z}^r \twoheadrightarrow \mathbb{Z},$$

where the last arrow denotes the summation $\mathbb{Z}^r \ni (a_1, \dots, a_m) \mapsto a_1 + \cdots + a_m \in \mathbb{Z}$.

Definition 3.4. *With the above notation, define $\Delta_C := \Delta_{\mathbb{C}^2 \setminus C_z, \varepsilon}$. This is called the (global) Alexander polynomial of the complex projective plane curve C.*

There is one more definition: The intersection of C_z with a very large 3-sphere yields also a link L_∞. The Alexander polynomial associated to L_∞ is called the Alexander polynomial of C at infinity and denoted by $\Delta_{C,\infty}$.

The connection between these three Alexander polynomials associated to a curve is given by the following theorem:

Theorem 3.5 (Libgober's divisibility theorems [23])**.** *Suppose that C is a complex projective plane curve of degree $d \geq 2$ with singular locus Σ. Then*

(a) $\Delta_C(t) \mid \prod_{p \in \Sigma} \Delta_{C,p}(t)$.

(b) $\Delta_C(t) \mid \Delta_{C,\infty}(t) = (t^d - 1)^{d-2}(t-1)$.

For cuspidal curves one has the following result:

Theorem 3.6. *Assume that C is a cuspidal curve of degree d with r components and κ cusps. Then*

$$\Delta_C(t) = \begin{cases} (t^2 - t + 1)^s & \textit{for some } s \in \{0, \dots, \kappa\} \quad \textit{if } C \textit{ is irreducible and } 6 \mid d, \\ (t-1)^{r-1} & \textit{otherwise.} \end{cases}$$

Proof: By Libgober's divisibility theorem and Lemma 3.3,

$$\Delta_C(t) \mid (t-1)^\delta (t^2 - t + 1)^\kappa,$$

where δ denotes the number of nodes of C. As both $t-1$ and $t^2 - t + 1$ are irreducible polynomials in $\mathbb{Q}[t]$, this implies $\Delta_C(t) = (t-1)^\rho (t^2 - t + 1)^s$ for some $\rho \in \mathbb{N}_0$ with $0 \leq \rho \leq \delta$

and some $s \in \mathbb{N}_0$ with $0 \leq s \leq \kappa$. By [26, Lemma 21], $\rho = r - 1$.

Assume now $s \geq 1$. Then C has at least one cusp, hence $d \geq 3$. By the second divisibility condition

$$(t-1)^{r-1}(t^2-t+1)^s \mid (t^d-1)^{d-2}(t-1).$$

In particular, any complex root of $t^2 - t + 1$ has to be a root of $(t^d-1)^{d-2}(t-1)$. Since the two roots of $t^2 - t + 1$ are the two primitive sixth roots of unity, this implies $6 \mid d$.

By [26, Theorem 34], $s = 0$ if C is not irreducible. □

The global Alexander polynomial $\Delta_C(t)$ is said to be trivial if it equals $(t-1)^{r-1}$, where r is the number of irreducible components of C. Theorem 3.6 states that the Alexander polynomial of an irreducible cuspidal curve is trivial if its degree is not divisible by six. The Alexander polynomial is trivial in even more cases, e. g. if $\pi_1(\mathbb{P}^2_{\mathbb{C}} \setminus C)$ is finite ([26, Corollary 30]) or abelian ([26, Lemma 36]).

3.4 Irregularity of cyclic multiple planes and the Mordell-Weil group revisited

Cyclic multiple planes ([23]). Let $C = V(f) \subset \mathbb{P}^2_{\mathbb{C}}$ be an irreducible cuspidal curve of degree d in the complex projective plane. Suppose that C intersects $V(z)$ transversely. For $m \in \mathbb{N}$ consider the surface

$$S_m := V(z^m - f(x,y,1)) \subset \mathbb{C}^3.$$

S_m is called a cyclic multiple plane. Note that

$$\psi_m : S_m \to \mathbb{C}^2, \quad (x,y,z) \mapsto (x,y)$$

is a ramified covering of $\mathbb{C}^2$ of degree m. The cardinality of a fiber $\psi_m^{-1}(p)$ for a point $p \in \mathbb{C}^2$ drops to one if and only if p lies on the affine plane curve defined by $f(x,y,1)$.

Let $\widetilde{S_m}$ be a desingularization of the projective closure of S_m in $\mathbb{P}^3_{\mathbb{C}}$. The irregularity of $\widetilde{S_m}$ is defined via $q(\widetilde{S_m}) := \dim_{\mathbb{Q}} H_1(\widetilde{S_m}, \mathbb{Q})$.

Proposition 3.7 (Libgober)**.** *Under the above hypotheses, suppose that the global Alexander polynomial of C is given by $\Delta_C(t) = (t^2 - t + 1)^s$ with $s \in \mathbb{N}_0$. Then $q(\widetilde{S_{6k}}) = 2s$ for any $k \in \mathbb{N}$.*

Proof: [23, Corollary 3.2] states that $q(\widetilde{S_m}) = \sum_i a_i$, where $a_i \in \mathbb{N}_0$ is the number of common roots of $t^{6k} - 1$ and the i-th irreducible factor in $\Delta_C(t)$. Since the roots of $t^2 - t + 1$ are precisely the two primitive sixth roots of unity, the claim follows. □

Already in 1931, Zariski proved:

Theorem 3.8 ([31]). *With the above notations, $q(\widetilde{S_m}) = 0$ unless m and d are both divisible by six. If $d = 6k$ for some $k \in \mathbb{N}$ and $6 \mid m$, then $q(\widetilde{S_m}) = 2 \dim_{\mathbb{C}} \operatorname{coker} \varphi$, where*

$$\varphi : \mathbb{C}[x, y, z]_{5k-3} \to \mathbb{C}^{\#\Sigma}, \quad f \mapsto (f(p))_{p \in \Sigma},$$

and Σ is the locus of cusps of C.

This map φ occured already in Theorem 2.12. Hence for irreducible curves, this gives immediately the following corollary:

Theorem 3.9. *Let $k \in \mathbb{N}$ and $f \in \mathbb{C}[x, y, z]_{6k}$ be an irreducible homogeneous polynomial of degree $6k$. Suppose that $C := V(f) \in \mathbb{P}^2_{\mathbb{C}}$ is a cuspidal curve intersecting $V(z)$ transversely. Then the following numbers coincide:*

(a) $2s$, *where* $\Delta_C(t) = (t^2 - t + 1)^s$,

(b) the irregularity of a resolution of singularities of the projective closure of S_6, where S_6 is the cyclic multiple plane $V(z^6 - f(x, y, 1)) \subset \mathbb{C}^3$,

(c) the $\mathbb{Z}$-rank of the Mordell-Weil group of an elliptic threefold birational to W_f,

(d) $2 \dim_{\mathbb{C}} \operatorname{coker} \varphi$, *where*

$$\varphi : \mathbb{C}[x, y, z]_{5k-3} \to \mathbb{C}^{\#\Sigma}, \quad f \mapsto (f(p))_{p \in \Sigma},$$

(e) $2 \dim_{\mathbb{C}} \operatorname{coker} \psi$, *where*

$$\psi : \mathbb{C}[x, y, z]_{7k-3} \to \mathbb{C}^{2\#\Sigma}, \quad f \mapsto \left(f(p), \frac{\partial f}{\partial \ell_p}(p) \right)_{p \in \Sigma},$$

and $\ell_p = 0$ is a reduced equation for the cuspidal tangent at $p \in \Sigma$.

The paper [24] gives another proof for (a) = (b) = (c). In the next chapter, the numbers (d) and (e) will be explored in terms of commutative algebra.

4 Ideals of cusps

For the whole chapter, define $S := \mathbb{C}[x, y, z]$. The locus of cusps of a cuspidal plane curve gives rise to three different ideals in S. The syzygies of these ideals are strongly related with the Alexander polynomial of the curve.

4.1 Some commutative algebra background

A reference for this section is e. g. [11].

Free resolutions. The ring S becomes a graded ring in a natural way by assigning to each of the variables x, y, z the degree one. Let $M = \bigoplus_{d\in\mathbb{Z}} M_d$ be a finitely generated graded S-module. A graded free resolution of M is an exact sequence of graded S-modules

$$\mathscr{F} : 0 \to F_3 \to F_2 \to F_1 \to F_0 \to M \to 0,$$

where either $F_i = 0$ or F_i is a free graded S-module of finite rank, $i = 0, 1, 2, 3$. Such a free module F_i may be written as a finite direct sum $F_i = \bigoplus_j S(-a_{ij})$ with $a_{ij} \in \mathbb{N}_0$ for all j. The numbers a_{ij} are unique if $\mathscr{F}$ is a minimal graded free resolution, i. e. if $\operatorname{im}(F_i) \subseteq \langle x, y, z\rangle F_{i-1}$ for $i = 1, 2, 3$.

By Hilbert's syzygy theorem (see [10, Corollary 19.7]), any finitely generated graded S-module M possesses a minimal graded free resolution. The number $\max\{i \mid F_i \neq 0\}$ is called the projective dimension of M and denoted by $\operatorname{pd} M$.

Hilbert function, Hilbert polynomial and Castelnuovo-Mumford regularity. The Hilbert function h_M of a finitely generated graded S-module M is defined via

$$h_M : \mathbb{Z} \to \mathbb{N}_0, \quad d \mapsto \dim_{\mathbb{C}} M_d.$$

Consider an exact sequence of finitely generated graded S-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0.$$

Then $h_{M_2}(d) = h_{M_1}(d) + h_{M_3}(d)$ for all $d \in \mathbb{Z}$.

For example, if $I, J \trianglelefteq S$ are homogeneous ideals such that $I \subseteq J$, then

$$0 \to J/I \to S/I \to S/J \to 0$$

is an exact sequence of finitely generated graded S-modules and $h_{J/I}(d) = h_{S/I}(d) - h_{S/J}(d)$. Now suppose that

$$\mathscr{F} : 0 \to \bigoplus_j S(-a_{3,j}) \to \bigoplus_j S(-a_{2,j}) \to \bigoplus_j S(-a_{1,j}) \to \bigoplus_j S(-a_{0,j}) \to M \to 0$$

is a graded free resolution of some graded S-module M. Then

$$h_M(d) = \sum_{i=0}^{3}(-1)^i \sum_{j:\, d \geq a_{ij}} \binom{d - a_{ij} + 2}{2}.$$

In particular, for $d \gg 0$, $h_M(d)$ coincides with

$$p_M(d) = \sum_{i=0}^{3}(-1)^i \sum_j \frac{(d - a_{ij} + 2)(d - a_{ij} + 1)}{2}.$$

p_M is called the Hilbert polynomial of M.

If $\mathscr{F}$ is minimal, the number $\mathrm{reg} M := \max_{i,j}\{a_{ij} - i\}$ is called the Castelnuovo-Mumford regularity of M. If $d \geq \mathrm{reg} M$, then $h_M(d) = p_M(d)$. The converse is true if M is a Cohen-Macaulay module.

4.2 Zero-dimensional subschemes of $\mathbb{P}^2_{\mathbb{C}}$

Projective subschemes of $\mathbb{P}^2_{\mathbb{C}}$ and saturated ideals. The complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ may be considered as the projective scheme

$$\operatorname{Proj} S := \{P \mid P \text{ a homogeneous prime ideal of } S \text{ such that } \langle x, y, z\rangle \not\subseteq P\}.$$

A projective subscheme of $\mathbb{P}^2_{\mathbb{C}}$ is given by $\operatorname{Proj} S/I$, where $I \trianglelefteq S$ is a homogeneous ideal.

If $I \trianglelefteq S$ is a homogeneous ideal, then its saturation I^{sat} is defined as

$$I^{\mathrm{sat}} := \bigcup_{m \in \mathbb{N}} I : \langle x, y, z\rangle^m = \{f \in S \mid \exists\, m \in \mathbb{N} : f \cdot \langle x, y, z\rangle^m \subseteq I\}.$$

Since S is noetherian and $I : \langle x, y, z\rangle^m \subseteq I : \langle x, y, z\rangle^n$ for $m \leq n$, I^{sat} is well-defined. Moreover, it is a homogeneous ideal and always $I \subseteq I^{\mathrm{sat}}$. An ideal I is called saturated if $I = I^{\mathrm{sat}}$. The geometric significance of saturated ideals is as follows: There is a 1-to-1 correspondence

$$\{\text{closed subschemes of } \mathbb{P}^2_{\mathbb{C}}\} \leftrightarrow \{\text{saturated homogeneous ideals in } \mathbb{C}[x, y, z]\}.$$

Two homogeneous ideals $I, J \trianglelefteq S$ define the same projective subscheme if and only if their saturations coincide ([19, Exercise II.5.10]).

Zero-dimensional subschemes of $\mathbb{P}^2_{\mathbb{C}}$. Let $I \trianglelefteq S$ be a saturated homogeneous ideal and $X := \operatorname{Proj} S/I$ the corresponding projective subscheme of $\mathbb{P}^2_{\mathbb{C}}$. The Hilbert function and the Hilbert polynomial of X are defined to be $h_X := h_{S/I}$ and $p_X := p_{S/I}$, respectively. X is called zero-dimensional if the Hilbert polynomial p_X is constant. In this case, the value of p_X is called the degree of X and denoted by $\deg X$. Note that $\deg X$ is in fact a natural number and may be interpreted as the number of points in X counted with appropriate multiplicities. In fact, Bézout's theorem (Theorem 2.1) may be rephrased as follows ([19, Exercise II.6.2]):

Theorem 2.1 reformulated. *Let C and C' be two complex projective plane curves without a common component. Then*

$$\deg C \cap C' = \deg C \cdot \deg C'.$$

Ideals of cusps as zero-dimensonial subschemes of $\mathbb{P}^2_{\mathbb{C}}$. Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a complex projective plane curve defined by the square-free homogeneous polynomial $f \in \mathbb{C}[x, y, z]$. Denote the singular locus of C by Σ and assume that $\Sigma \neq \emptyset$. Then one can associate the following three ideals to Σ:

(a) $I(\Sigma)$, the homogeneous radical ideal defining Σ as a Zariski-closed subset of $\mathbb{P}^2_{\mathbb{C}}$,

(b) $J(f) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$, the Jacobian ideal of f,

(c) $J(f)^{\text{sat}}$, the saturation of $J(f)$.

All three ideals are homogeneous and have the same vanishing set, namely Σ. Moreover $J(f) \subseteq J(f)^{\text{sat}} \subseteq I(\Sigma)$. But $I(\Sigma)$ and $J(f)^{\text{sat}}$ define different projective subschemes. The reason is that $I(\Sigma)$ induces the so-called reduced scheme structure, while $J(f)^{\text{sat}}$ puts more geometry into the scheme structure:

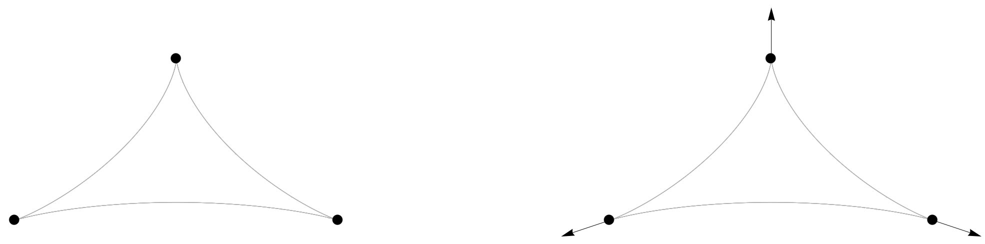
Example: Consider the cuspidal cubic $C = V(f)$, where $f := x^2z - y^3 \in S_3$. Its singular locus is $\Sigma := \{(0 : 0 : 1)\}$. Then

$$I(\Sigma) = \langle x, y \rangle, \quad J(f) = \langle xz, y^2, x^2 \rangle, \quad J(f)^{\text{sat}} = \langle x, y^2 \rangle.$$

For example, the line $V(y)$ passes through Σ and $y \notin J(f)^{\text{sat}}$. On the other hand, $V(y^2)$ does also pass through Σ, but $y^2 \in J(f)^{\text{sat}}$. Observe that $\frac{\partial}{\partial y} y^2 = 2y$ vanishes at p, whereas $\frac{\partial}{\partial y} y = 1$ does not. Since $\frac{\partial}{\partial y}$ is the directional derivative in the direction of the cuspidal tangent $V(x)$ at $(0 : 0 : 1)$, this means visually that $J(f)^{\text{sat}}$ contains only polynomials that respect the cuspidal tangent appropriately.

Lemma 4.1. *Let $C = V(f)$ be a complex projective plane curve such that all singular points are ordinary cusps. Denote by Σ the locus of cusps of C. Then*

$$J(f)^{\text{sat}} = \left\langle g \in S \text{ homogeneous} \;\middle|\; g(p) = 0 \text{ and } \frac{\partial g}{\partial \ell_p}(p) = 0 \text{ for all } p \in \Sigma \right\rangle.$$



Proj $S/I(\Sigma)$ versus Proj $S/J(f)^{\mathrm{sat}}$

Proof: Call the right-hand side K.

- K is a saturated ideal: Let $g \in \mathbb{C}[x,y,z]$ be a homogeneous polynomial satisfying $g \cdot \langle x,y,z\rangle^m \subseteq K$ for some $m \in \mathbb{N}$. Fix a cusp $p = (\alpha : \beta : 1)$. Then $0 = (gz^m)(p) = g(p)$. Moreover
$$0 = \frac{\partial(gz^m)}{\partial \ell_p}(p) = \frac{\partial z^m}{\partial \ell_p}(p) \cdot g(p) + 1^m \cdot \frac{\partial g}{\partial \ell_p}(p) = \frac{\partial g}{\partial \ell_p}(p).$$
Consequently $g \in K$.

- Let $\kappa := \#\Sigma$ and decompose
$$J(f)^{\mathrm{sat}} = Q_1 \cap \cdots \cap Q_\kappa, \quad K = Q_1' \cap \cdots \cap Q_\kappa'$$
into an intersection of primary ideals, where $\sqrt{Q_i} = \sqrt{Q_i'}$ is the prime ideal corresponding to the i-th cusp of C, $i = 1, \ldots, \kappa$. Suppose that p is the cusp corresponding to Q_i resp. Q_i'. Then, after a suitable coordinate change, $p = (0:0:1)$, $f(x,y,1) = x^2 - y^3$ and $\ell_p = V(x)$. Now one sees that $Q_i = \langle x, y^2 \rangle = Q_i'$. □

If C is a cuspidal curve that has also nodes as singularities, decompose
$$J(f)^{\mathrm{sat}} = \bigcap_{p \text{ cusp of } C} Q_p \quad \cap \bigcap_{q \text{ node of } C} Q_q,$$
where Q_p resp. Q_q is a primary ideal with respect to the prime ideal at p resp. q. Then Lemma 4.1 still holds true when $J(f)^{\mathrm{sat}}$ is replaced by $\bigcap_{p \text{ cusp of } C} Q_p$.

4.3 Codimension two ideals in $\mathbb{C}[x,y,z]$

All three ideals of cusps mentioned in the previous section have codimension two, as they define zero-dimensional subschemes of $\mathbb{P}^2_{\mathbb{C}}$. The following section explores the properties of codimension two ideals of S in general. The main ideas for this section are taken from [11, Chapter 3].

Projective dimension.

Lemma 4.2. *Let $I \trianglelefteq S$ be a homogeneous ideal of codimension 2. Then the projective dimension of the graded S-module S/I is either 2 or 3.*

Proof: Let M be a graded S-module. Then $\operatorname{depth} M$ is defined to be the length of a maximal regular sequence in M with respect to the maximal ideal $\langle x,y,z\rangle \trianglelefteq S$. Clearly $\operatorname{depth} S = 3$, as (x,y,z) is a maximal regular sequence. Using the Auslander-Buchsbaum formula [10, Theorem 19.9] and the fact that $\operatorname{depth} S/I \leq \dim S/I$ [10, Proposition 18.2], one obtains

$$\operatorname{pd} S/I = \operatorname{depth} S - \operatorname{depth} S/I \geq \operatorname{depth} S - \dim S/I = 3 - 1 = 2.$$

Hilbert's syzygy theorem [10, Corollary 19.7] shows that $\operatorname{pd} S/I \leq 3$. □

The following lemma (inspired by [4, Proposition 5.2]) provides some tools to compute the projective dimension of a codimension two ideal.

Lemma 4.3. *Let $I \trianglelefteq S$ be a homogeneous ideal of codimension 2. Then the following are equivalent:*

(a) $\operatorname{pd} S/I = 2$,

(b) S/I *is a Cohen-Macaulay module over* S,

(c) $\langle x,y,z\rangle$ *is not an associated prime of* S/I,

(d) $I : \langle x,y,z\rangle = I$,

(e) I *is a saturated ideal.*

Proof:

$(a) \Leftrightarrow (b)$: S/I is Cohen-Macaulay if and only if $\operatorname{depth} S/I = \dim S/I$, i. e. the bound in the proof of the previous lemma is sharp.

$(a) \Leftrightarrow (c)$: The ring S is a Cohen-Macaulay ring, $\operatorname{depth} S = \dim S = 3$. Thus the graded version of [10, Corollary 19.10] implies that $\operatorname{pd} S/I = 3$ if and only if $\langle x,y,z\rangle$ is an associated prime of S/I. By Lemma 4.2, $\operatorname{pd} S/I \in \{2,3\}$.

$(c) \Leftrightarrow (d)$: If $\langle x,y,z\rangle$ is an associated prime of S/I, then there is some $f \in S$ such that $\langle x,y,z\rangle = A(f) := \{g \in S \mid fg \in I\}$. In particular $f \notin I$, because otherwise $A(f) = S$. Moreover $f \in I : \langle x,y,z\rangle = \{h \in S \mid h\langle x,y,z\rangle \subseteq I\}$.
Conversely, if there exists some $f \in (I : \langle x,y,z\rangle) \setminus I$, then $A(f) \supseteq \langle x,y,z\rangle$. Since $f \cdot 1 \notin I$, the ideal $A(f)$ is not the whole ring S. As $\langle x,y,z\rangle \trianglelefteq S$ is a maximal ideal, this shows $A(f) = \langle x,y,z\rangle$. Hence $\langle x,y,z\rangle$ is associated to S/I.

$(d) \Leftrightarrow (e)$: This is clear by the definition of saturation. □

Syzygies and regularity in the case $\operatorname{pd} S/I = 2$.

Lemma 4.4. *If $I \trianglelefteq S$ is a codimension two ideal with $\operatorname{pd} S/I = 2$, then a minimal graded free resolution of the S-module S/I is of the form*

$$0 \to \bigoplus_{i=1}^{t} S(-b_i) \to \bigoplus_{i=1}^{t+1} S(-a_i) \to S \to S/I \to 0$$

with the properties

(a) $t, a_1, \dots, a_{t+1}, b_1, \dots, b_t \in \mathbb{N}$,

(b) $b_i \geq a_i + 1 \geq a_{i+1} + 1$ *for* $i = 1, \dots, t$, $a_1 \geq \dots \geq a_{t+1}$ *and* $b_1 \geq \dots \geq b_t$.

(c) $\sum_{i=1}^{t+1} a_i = \sum_{i=1}^{t} b_i$,

(d) $\sum_{i=1}^{t} b_i^2 - \sum_{i=1}^{t+1} a_i^2 = 2 \deg \operatorname{Proj} S/I$.

Proof: Since S/I has projective dimension two, the Hilbert-Burch Theorem ([11, Theorem 3.2]) states that a minimal graded free resolution of S/I is of the form

$$0 \to F \to G \to S \to S/I \to 0,$$

where F and G are free graded S-modules of finite rank such that $\operatorname{rk} G = \operatorname{rk} F + 1$. Let $t := \operatorname{rk} F \in \mathbb{N}$. Write $F = \bigoplus_{i=1}^{t} S(-b_i)$ and $G = \bigoplus_{i=1}^{t+1} S(-a_i)$ for natural numbers $a_1 \geq \dots \geq a_{t+1}$ and $b_1 \geq \dots \geq b_t$. By [11, Proposition 3.7], $b_i \geq a_i + 1 \geq a_{i+1} + 1$ for $i = 1, \dots, t$, and moreover

$$\sum_{i=1}^{t+1} a_i = \sum_{i=1}^{t} b_i.$$

Since $\operatorname{Proj} S/I$ is zero-dimensional, the Hilbert polynomial $p_{S/I}$ is constant. Consequently

$$\begin{aligned}
\deg \operatorname{Proj} S/I &= p_{S/I}(0) \\
&= \binom{2}{2} - \sum_{i=1}^{t+1} \frac{(-a_i+2)(-a_i+1)}{2} + \sum_{i=1}^{t} \frac{(-b_i+2)(-b_i+1)}{2} \\
&= 1 + \frac{1}{2}\left(\sum_{i=1}^{t} (b_i^2 - 3b_i + 2) - \sum_{i=1}^{t+1} (a_i^2 - 3a_i + 2) \right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{t} b_i^2 - \sum_{i=1}^{t+1} a_i^2 \right) - \frac{3}{2}\left(\sum_{i=1}^{t} b_i - \sum_{i=1}^{t+1} a_i \right) + 1 + t - (t+1) \\
&= \frac{1}{2}\left(\sum_{i=1}^{t} b_i^2 - \sum_{i=1}^{t+1} a_i^2 \right).
\end{aligned}$$

□

The following technical proposition gives a bound on the numbers appearing in a minimal resolution of S/I.

Proposition 4.5. *Let $I \trianglelefteq S$ be a codimension two ideal with $\operatorname{pd} S/I = 2$. Suppose further that there exists some $r \in \mathbb{Q}$, $r \geq 3$ such that*

$$\forall n_0 \in \mathbb{N} \quad \exists n \geq n_0 : \quad rn \in \mathbb{N} \quad \text{and} \quad p_{S/\varphi_n^*(I)}(rn-3) - h_{S/\varphi_n^*(I)}(rn-3) \leq rn,$$

where φ_n is the Kummer covering from Chapter 2. Then

$$h_{S/I}(m) = p_{S/I}(m) \quad \text{for } m > r-3.$$

Moreover, if

$$0 \to \bigoplus_{i=1}^{t} S(-b_i) \to \bigoplus_{i=1}^{t+1} S(-a_i) \to S \to S/I \to 0$$

is a minimal graded free resolution of S/I, then $a_i \leq r-1$ for $i = 1, \dots, t+1$ and $b_i \leq r$ for $i = 1, \dots, t$. Furthermore, if $r \in \mathbb{N}$, then

$$p_{S/I}(r-3) - h_{S/I}(r-3) = \#\{i \in \{1, \dots, t\} \mid b_i = r\}.$$

Proof: The proof is analogous to the proof of [21, Proposition 3.3].

- By Lemma 4.4, I has a minimal graded free resolution in the described form. The Hilbert function of S/I evaluated at $m \in \mathbb{N}_0$ is

$$h_{S/I}(m) = \binom{m+2}{2} - \sum_{i:a_i \leq m} \binom{m-a_i+2}{2} + \sum_{i:b_i \leq m} \binom{m-b_i+2}{2}.$$

  The Hilbert polynomial is given by

$$p_{S/I}(m) = \binom{m+2}{2} - \sum_{i=1}^{t+1} \frac{(m-a_i+2)(m-a_i+1)}{2} + \sum_{i=1}^{t} \frac{(m-b_i+2)(m-b_i+1)}{2}.$$

  This implies that

$$\begin{aligned} &p_{S/I}(m) - h_{S/I}(m) \\ &\quad = \sum_{i:b_i \geq m+1} \frac{(m-b_i+2)(m-b_i+1)}{2} - \sum_{i:a_i \geq m+1} \frac{(m-a_i+2)(m-a_i+1)}{2} \\ &\quad = \sum_{i:b_i \geq m+3} \frac{(m-b_i+2)(m-b_i+1)}{2} - \sum_{i:a_i \geq m+3} \frac{(m-a_i+2)(m-a_i+1)}{2}. \end{aligned}$$

- Assume now that $b_i > r$ for some i. Choose a number $n_0 \in \mathbb{N}$ such that

$$(*) \quad \begin{cases} (n_0(b_i-r)+2)(n_0(b_i-r)+1) & > 2rn_0 \quad \text{if } a_i < r, \\ (n_0(b_i-r)+2)(n_0(b_i-r)+1) & \\ \quad -(n_0(a_i-r)+2)(n_0(a_i-r)+1) & > 2rn_0 \quad \text{if } a_i \geq r. \end{cases}$$

Select $n \geq n_0$ such that $rn \in \mathbb{N}$ and $p_{S/\varphi_n^*(I)}(rn-3) - h_{S/\varphi_n^*(I)}(rn-3) \leq rn$. Then n is still satisfying $(*)$ when n_0 is replaced by n.

A minimal graded free resolution of $S/\varphi_n^*(I)$ is now given by

$$0 \to \bigoplus_{i=1}^{t} S(-nb_i) \to \bigoplus_{i=1}^{t+1} S(-na_i) \to S \to S/\varphi_n^*(I) \to 0.$$

Thus, as above,

$$\begin{aligned}
&p_{S/\varphi_n^*(I)}(rn-3) - h_{S/\varphi_n^*(I)}(rn-3) \\
&\quad = \sum_{i:b_i \geq r} \frac{(n(r-b_i)-1)(n(r-b_i)-2)}{2} - \sum_{i:a_i \geq r} \frac{(n(r-a_i)-1)(n(r-a_i)-2)}{2} \\
&\quad = \sum_{i:b_i \geq r} \frac{(n(b_i-r)+1)(n(b_i-r)+2)}{2} - \sum_{i:a_i \geq r} \frac{(n(a_i-r)+1)(n(a_i-r)+2)}{2} \\
&\quad \overset{(*)}{>} rn.
\end{aligned}$$

This contradicts the hypothesis $p_{S/\varphi_n^*(I)}(rn-3) - h_{S/\varphi_n^*(I)}(rn-3) \leq rn$.

Consequently, $b_i \leq r$ for $i = 1, \ldots, t$. Due to the properties of the resolution of S/I (see Lemma 4.4), $a_i \leq b_i - 1 \leq r-1$ for $i = 1, \ldots, t$ and $a_{t+1} \leq a_t \leq b_t - 1 \leq r-1$.

- If $r \in \mathbb{N}$, then the bounds $a_i \leq r-1$ and $b_i \leq r$ give

$$\begin{aligned}
&p_{S/I}(r-3) - h_{S/I}(r-3) \\
&\quad = \sum_{i:b_i \geq r} \frac{(r-b_i-1)(r-b_i-2)}{2} - \sum_{i:a_i \geq r} \frac{(r-a_i-1)(r-a_i-2)}{2} \\
&\quad = \#\{i \in \{1, \ldots, t\} \mid b_i = r\}.
\end{aligned}$$

- Finally let $m > r-3$. Then

$$\begin{aligned}
&p_{S/I}(m) - h_{S/I}(m) \\
&\quad = \sum_{i:b_i > r} \frac{(m-b_i+2)(m-b_i+1)}{2} - \sum_{i:a_i > r} \frac{(m-a_i+2)(m-a_i+1)}{2} \\
&\quad = 0,
\end{aligned}$$

which finishes the proof. $\square$

Since $\operatorname{pd} S/I = 2$ means that S/I is a Cohen-Macaulay module, this proposition also gives the Castelnuovo-Mumford regularity if at least one b_i is equal to r.

4.4 Ideals of cusps and the Alexander polynomial

We will now apply the results on codimension two ideals in S to the three ideals of cusps associated to a cuspidal curve C. This will establish a connection to the global Alexander polynomial of Δ_C. Throughout this section, C is assumed to be a singular curve having only ordinary cusps as singularities. The set of cusps of C will be denoted by Σ.

A first result is:

Corollary 4.6. *Let $C = V(f)$ be of degree $6k$ for some $k \in \mathbb{N}$. Then the following numbers are equal:*

(a) $\frac{1}{2}\operatorname{rk}\mathrm{MW}(X_f)$, *where X_f is an associated elliptic threefold to C,*

(b) $\#\Sigma - h_{S/I(\Sigma)}(5k-3)$,

(c) $2\#\Sigma - h_{S/J(f)^{\mathrm{sat}}}(7k-3)$.

Proof: By Theorem 2.12, $\frac{1}{2}\operatorname{rk}\mathrm{MW}(X_f) = \dim_{\mathbb{C}}\operatorname{coker}\varphi$, where

$$\varphi : S_{5k-3} \to \mathbb{C}^{\#\Sigma}, \quad f \mapsto (f(p))_{p\in\Sigma}.$$

Now the dimension of $\operatorname{coker}\varphi$ can be computed as follows:

$$\begin{aligned}\dim_{\mathbb{C}}\operatorname{coker}\varphi &= \dim_{\mathbb{C}}\mathbb{C}^{\#\Sigma} - \dim_{\mathbb{C}} S_{5k-3} + \dim_{\mathbb{C}}\ker\varphi \\ &= \#\Sigma - \dim_{\mathbb{C}} S_{5k-3} + \dim_{\mathbb{C}} I(\Sigma)_{5k-3} \\ &= \#\Sigma - h_{S/I(\Sigma)}(5k-3).\end{aligned}$$

Using Lemma 4.1, (c) follows in a similar way. □

Ideals of cusps and Kummer coverings. The following lemma investigates the effect of the Kummer covering φ_n on the ideal of cusps and on the saturation of the Jacobian ideal.

Lemma 4.7. *Let $C = V(f)$ satisfy the hypotheses of Proposition 2.8, i. e. C intersects $V(xyz)$ in smooth points only and C does not contain any of the points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$. Then for $n \in \mathbb{N}$, the morphism φ_n satisfies*

(a) $I(\varphi_n^{-1}(\Sigma)) = \varphi_n^*(I(\Sigma))$,

(b) $J(\varphi_n^*(f))^{\mathrm{sat}} = \varphi_n^*(J(f)^{\mathrm{sat}})$.

Proof:

(a) The vanishing set of the ideal $\varphi_n^*(I(\Sigma))$ is precisely $\varphi_n^{-1}(\Sigma)$. Thus in view of Hilbert's Nullstellensatz, it suffices to show that $\varphi_n^*(I(\Sigma))$ is a radical ideal. Since no point of Σ lies on $V(z)$, the radical ideal $I(\Sigma)$ is the homogenization of some radical ideal $I \trianglelefteq \mathbb{C}[x,y]$. Observe that φ_n^* satisfies

$$\varphi_n^*(I(\Sigma)) = \varphi_n^*(I^{\mathrm{hom}}) = \varphi_n^*(I)^{\mathrm{hom}},$$

where $(\cdot)^{\text{hom}}$ denotes the homogenization. Because homogenizations of radical ideals are radical, it is enough to show that $\varphi_n^*(I)$ is a radical ideal. The latter ideal is zero-dimensional and therefore [16, Proposition 4.5.1] yields

$$\sqrt{\varphi_n^*(I)} = \varphi_n^*(I) + \langle g_x', g_y' \rangle,$$

where $\varphi_n^*(I) \cap \mathbb{C}[x] = \langle g_x \rangle$ and g_x' is the square-free part of g_x (analogously for g_y). That is, g_x' is the square-free part of $\prod(x^n - \alpha) \in \varphi_n^*(I)$, where the product is taken over all $\alpha \in \mathbb{C}^*$ such that there exists some $\beta \in \mathbb{C}^*$ with $(\alpha, \beta) \in V(I) \subset \mathbb{C}^2$. But this polynomial has no multiple factor, thus $g_x' = g_x \in \varphi_n^*(I)$. By the same reasoning, $g_y' \in \varphi_n^*(I)$. Hence $\varphi_n^*(I)$ is a radical ideal, and so is $\varphi_n^*(I(\Sigma))$.

(b) Since both ideals define the same subscheme of $\mathbb{P}^2_{\mathbb{C}}$, it suffices to show that $\varphi_n^*(J(f)^{\text{sat}})$ is saturated. The ideal $J(f)^{\text{sat}}$ is saturated and of codimension two, hence $S/J(f)^{\text{sat}}$ has projective dimension 2 by Lemma 4.3. Following Lemma 4.4, a minimal graded free resolution of $S/J(f)^{\text{sat}}$ is of the form

$$0 \to \bigoplus_{i=1}^{t} S(-b_i) \to \bigoplus_{i=1}^{t+1} S(-a_i) \to S \to S/J(f)^{\text{sat}} \to 0.$$

Consequently

$$0 \to \bigoplus_{i=1}^{t} S(-nb_i) \to \bigoplus_{i=1}^{t+1} S(-na_i) \to S \to S/\varphi_n^*(J(f)^{\text{sat}}) \to 0$$

is a graded free resolution of $\varphi_n^*(J(f)^{\text{sat}})$. Therefore $\operatorname{pd} S/\varphi_n^*(J(f)^{\text{sat}}) \leq 2$. Since $\varphi_n^*(J(f)^{\text{sat}})$ is an ideal of codimension two, this shows by Lemma 4.2 and Lemma 4.3 that $\varphi_n^*(J(f)^{\text{sat}})$ is indeed a saturated ideal. $\square$

Syzygies of $I(\Sigma)$ and $J(f)^{\text{sat}}$. In the sequel, Proposition 4.5 and Corollary 4.6 are used to obtain upper bounds on the degrees of the syzygies of $S/I(\Sigma)$ and $S/J(f)^{\text{sat}}$.

Corollary 4.8. *Let $C = V(f)$ be irreducible of degree d and define $I := I(\Sigma)$. Then S/I has a minimal graded free resolution as in Lemma 4.4 such that $a_i \leq \frac{5}{6}d - 1$ and $b_i \leq \frac{5}{6}d$ for all i. Moreover*

$$\#\Sigma - h_{S/I}(m) = \begin{cases} \#\{i \mid b_i = \frac{5}{6}d\} & \textit{if } m = \frac{5}{6}d - 3 \textit{ and } 6 \mid d, \\ 0 & \textit{if } m > \frac{5}{6}d - 3. \end{cases}$$

In particular, $\Delta_C(t) = (t^2 - t + 1)^s$, where $s = \#\{i \mid b_i = \frac{5}{6}d\}$.

Proof: The maximal ideal $\langle x, y, z \rangle$ is not associated to S/I, hence $\operatorname{pd} S/I = 2$. Consequently, S/I has a minimal graded free resolution in the desired shape by Lemma 4.3.

The Hilbert polynomial of S/I is constant, namely $p_{S/I}(m) = \#\Sigma$ for all m. Fix $n_0 \in \mathbb{N}$ and define $n := 6n_0$. If necessary, perform a projective change of coordinates such that C satisfies the prerequisites of Proposition 2.8 with $\Delta = \emptyset$. Then the preimage $\varphi_n^{-1}(C)$ is a cuspidal curve of degree $dn = 6dn_0$, whose ideal of cusps is given by $\varphi_n^*(I(\Sigma))$ according to Lemma 4.7. By Corollary 4.6 and Lemma 2.11,

$$p_{S/\varphi_n^*(I)}(5dn_0 - 3) - h_{S/\varphi_n^*(I)}(5dn_0 - 3) = \frac{1}{2} \operatorname{rk} \operatorname{MW}(X_{\varphi_n^*(f)}) \leq 5dn_0 - 1,$$

where $X_{\varphi_n^*(f)}$ is an elliptic threefold associated to $\varphi_n^{-1}(C)$. Thus Proposition 4.5 applies with $n = 6n_0$ and $r = \frac{5}{6}d$.

For the "in particular" statement, assume first that $d = 6k$ for some $k \in \mathbb{N}$. Theorem 3.6 guarantees $\Delta_C(t) = (t^2 - t + 1)^s$ for some $s \in \mathbb{N}_0$. Now

$$s = \frac{1}{2} \operatorname{rk} \operatorname{MW}(X_f) = p_{S/I}(5k - 3) - h_{S/I}(5k - 3) = \#\{i \mid b_i = 5k\}$$

in virtue of Theorem 3.9. If d is not divisible by six, then $\Delta_C(t) = 1$ again by Theorem 3.6 and obviously $\#\{i \mid b_i = \frac{5}{6}d\} = 0$. □

For the saturation of the Jacobian, one has:

Corollary 4.9. *Let $C = V(f)$ be irreducible of degree d and define $J := J(f)^{\text{sat}}$. Then S/J has a minimal graded free resolution of the form as in Lemma 4.4 such that $a_i \leq \frac{7}{6}d - 1$ and $b_i \leq \frac{7}{6}d$ for all i. Moreover*

$$2\#\Sigma - h_{S/J^{\text{sat}}}(m) = \begin{cases} \#\{i \mid b_i = \frac{7}{6}d\} & \text{if } m = \frac{7}{6}d - 3 \text{ and } 6 \mid d, \\ 0 & \text{if } m > \frac{7}{6}d - 3. \end{cases}$$

In particular, $\Delta_C(t) = (t^2 - t + 1)^s$, where $s = \#\{i \mid b_i = \frac{7}{6}d\}$.

Proof: Since J is saturated, Lemma 4.3 yields $\operatorname{pd} S/J = 2$ and Lemma 4.4 applies.

The Hilbert polynomial of S/J is constantly $2\#\Sigma$, as each cusp counts with multiplicity two: If $p = (\alpha : \beta : 1)$, then $\frac{\partial f}{\partial x}(x, y, 1)$ and $\frac{\partial f}{\partial y}(x, y, 1)$ intersect with multiplicity two at p as shown in the proof of Lemma 2.7. This intersection multiplicity and the degree of the component of $\operatorname{Proj} S/J$ located at p are equal, as can be seen by comparing the two versions of Bézout's theorem.

Now the proof goes along the same lines as for the previous corollary: Fix $n_0 \in \mathbb{N}$ and let $n := 6n_0$. After a suitable coordinate change, the ideal $\varphi_n^*(J)$ is by Lemma 4.7 the saturation of the Jacobian ideal of the cuspidal curve $\varphi_n^{-1}(C)$ of degree $dn = 6dn_0$. Due to Corollary 4.6 and Lemma 2.11,

$$p_{S/\varphi_n^*(J)}(7dn_0 - 3) - h_{S/\varphi_n^*(J)}(7dn_0 - 3) = \frac{1}{2} \operatorname{rk} \operatorname{MW}(X_{\varphi_n^*(f)}) \leq 5dn_0 - 1 \leq 7dn_0.$$

Now use Proposition 4.5 with $n = 6n_0$ and $r = \frac{7}{6}d$ to obtain the assertion. The "in particular" statement can be proven as in Corollory 4.8. □

One can also give a lower bound on the degree of the syzygies ([21, Proposition 2.5]): Let

$$0 \to \bigoplus_{i=1}^{t} S(-b_i) \to \bigoplus_{i=1}^{t+1} S(-a_i) \to S \to S/I \to 0$$

be a minimal graded free resolution of S/I, where $I := I(\Sigma)$. Suppose that $a_1 \geq \cdots \geq a_{t+1}$. Then all cusps of C lie on a curve of degree a_{t+1}, say C'. Define $\kappa := \#\Sigma$. Bézout's theorem 2.1 yields $2\kappa \leq a_{t+1}d$ or C and C' have a common irreducible component C'' of degree $d'' \leq \min(a_{t+1}, d)$. In the latter case, as cusps are irreducible singularities, all cusps of C are necessarily cusps of C''. The number of cusps of C'' is bounded by $d''(d''-2)/3$ due to Proposition 2.5. Hence

$$\frac{(d'')^2}{2} \leq \frac{a_{t+1}d}{2} \leq \kappa \leq \frac{d''(d''-2)}{3},$$

which implies $d'' = 0$. Consequently, C and C' have no common irreducible component and

$$a_{t+1} \geq \frac{2\kappa}{d}.$$

Let J denote the saturation of the Jacobian ideal of f. If g is a generator of minimal degree, then $g \in I$, as $J \subseteq I$. Thus $\deg g \geq 2\kappa/d$. This shows the following lemma:

Lemma 4.10. *Let C be a cuspidal curve of degree d with κ cusps. If*

$$0 \to \bigoplus_{i=1}^{t} S(-b_i) \to \bigoplus_{i=1}^{t+1} S(-a_i) \to S \to S/I \to 0$$

is a minimal graded free resolution of I, where I is either the locus of cusps of C or the saturation of the Jacobian ideal of C, then $a_i \geq 2\kappa/d$, $i = 1, \ldots, t+1$ and $b_i \geq 2\kappa/d + 1$, $i = 1, \ldots, t$.

Syzygies of the Jacobian ideal. One can now compute the Alexander polynomial of an irreducible cuspidal curve from a minimal graded free resolution or the Hilbert function of S/I, where I is either $I(\Sigma)$ or $J(f)^{\mathrm{sat}}$. Furthermore, it is also possible to use the Jacobian ideal $J(f)$ directly.

Lemma 4.11. *Let C be of degree d with Jacobian ideal J. Assume that J is not saturated. A minimal graded free resolution of S/J is of the form*

$$0 \to \bigoplus_{i=1}^{t} S(-c_i) \to \bigoplus_{i=1}^{t+2} S(-b_i) \to S(-d+1)^3 \to S \to S/J \to 0$$

with the properties

(a) $t, b_1, \ldots, b_{t+2}, c_1, \ldots, c_t \in \mathbb{N}$.

(b) $c_i \geq b_i + 1$ *for* $i = 1, \ldots, t$, $c_1 \geq \cdots \geq c_t$ *and* $b_1 \geq \cdots \geq b_{t+2}$.

Proof: Since J is not saturated, $\mathrm{pd}(S/J) = 3$ by Lemma 4.3. Now [20, Lemma 1.1] applies. □

The following is the analogue of Corollary 4.8 and Corollary 4.9.

Proposition 4.12. *Let C be irreducible of degree d with Jacobian ideal J. Suppose that S/J has a minimal graded free resolution as in Lemma 4.11. Then*

$$\begin{aligned} 3d-4 \geq b_i \geq \frac{11}{6}d-3 \quad &\textit{for } i=1,\dots,t+2, \\ 3d-3 \geq c_i \geq \frac{11}{6}d-2 \quad &\textit{for } i=1,\dots,t. \end{aligned}$$

Moreover, $\Delta_C(t) = (t^2-t+1)^s$*, where* $s = \#\{i \in \{1,\dots,t+2\} \mid b_i = \frac{11}{6}d-3\}$*.*

Proof:

- [20, Theorem 1.3 (i)] states that the graded S-module J^{sat}/J has the minimal graded free resolution

$$0 \to \bigoplus_{i=1}^{t} S(-c_i) \to \bigoplus_{i=1}^{t+2} S(-b_i) \to \bigoplus_{i=1}^{t+2} S(b_i-3d+3) \to \bigoplus_{i=1}^{t} S(c_i-3d+3) \to J^{\text{sat}}/J \to 0$$

  if $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are linearly independent. Assume the contrary, say $\frac{\partial f}{\partial z}$ is some linear combination of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$. Then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ have precisely κ common roots, where κ is the number of cusps of C. Applying Bézout's theorem yields $2\kappa = (d-1)^2$, which is impossible by Proposition 2.5.

- Suppose that there is some $i \in \{1,\dots,t+2\}$ such that $b_i < \frac{11}{6}d-3$. Then $3d-3-b_i > \frac{7}{6}d$, i. e. J^{sat}/J has a syzygy of degree $> \frac{7}{6}d$. This contradicts Corollary 4.9. Due to Lemma 4.11, $c_i \geq b_i+1 \geq \frac{11}{6}d-2$ for all $i=1,\dots,t$. The bound $c_i \leq 3d-3$ for all i is obvious in view of the resolution of J^{sat}/J, and thus $b_{t+2} \leq b_{t+1} \leq b_i \leq c_i - 1 \leq 3d-4$ for all $i=1,\dots,t$.

- The Hilbert function of J^{sat}/J is symmetric: Let $m \in \mathbb{Z}$. Then

$$\begin{aligned} h_{J^{\text{sat}}/J}(m) &= \sum_{i=1}^{t} \binom{m+2+(c_i-3d+3)}{2} - \sum_{i=1}^{t+2} \binom{m+2+(b_i-3d+3)}{2} \\ &\quad + \sum_{i=1}^{t+2} \binom{m+2-b_i}{2} - \sum_{i=1}^{t} \binom{m+2-c_i}{2} \\ &= \sum_{i=1}^{t} \left(\binom{m-3d+5+c_i}{2} - \binom{m+2-c_i}{2} \right) \\ &\quad + \sum_{i=1}^{t+2} \left(\binom{m+2-b_i}{2} - \binom{m-3d+5-b_i}{2} \right) \\ &= \sum_{i=1}^{t} \left(\binom{(3d-6-m)+2-c_i}{2} - \binom{(3d-6-m)-3d-5+c_i}{2} \right) \\ &\quad + \sum_{i=1}^{t+2} \left(\binom{(3d-6-m)-3d-5+b_i}{2} - \binom{(3d-6-m)+2-b_i}{2} \right) \\ &= h_{J^{\text{sat}}/J}(3d-6-m). \end{aligned}$$

- Assume now that $d = 6k$. Since $h_{J^{\mathrm{sat}}/J}(m) = h_{S/J}(m) - h_{S/J^{\mathrm{sat}}}(m)$ for any $m \in \mathbb{Z}$, this implies

$$h_{S/J}(7k-3) - h_{S/J^{\mathrm{sat}}}(7k-3) = h_{S/J}(11k-3) - h_{S/J^{\mathrm{sat}}}(11k-3).$$

If $\Delta_C(t) = (t^2 - t + 1)^s$, then by Corollary 4.9

$$h_{S/J^{\mathrm{sat}}}(11k-3) - h_{S/J^{\mathrm{sat}}}(7k-3) = 2\#\Sigma - h_{S/J^{\mathrm{sat}}}(7k-3) = s,$$

thus

$$\begin{aligned} s &= h_{S/J}(11k-3) - h_{S/J}(7k-3) \\ &= \binom{11k-1}{2} - 3\binom{5k}{2} + \sum_{i=1}^{t+2}\binom{11k-1-b_i}{2} - \sum_{i=1}^{t}\binom{11k-1-c_i}{2} \\ &\quad - \binom{7k-1}{2} + 3\binom{k}{2} - \sum_{i=1}^{t+2}\binom{7k-1-b_i}{2} + \sum_{i=1}^{t}\binom{7k-1-c_i}{2} \\ &= \sum_{i=1}^{t+2}\binom{11k-1-b_i}{2} - \sum_{i=1}^{t}\binom{11k-1-c_i}{2} \\ &\quad - \sum_{i=1}^{t+2}\binom{7k-1-b_i}{2} + \sum_{i=1}^{t}\binom{7k-1-c_i}{2}. \end{aligned}$$

Since $c_i \geq 11k-2$ and $b_i \geq 11k-3$ for all i,

$$s = \#\{i \in \{1, \ldots, t+2\} \mid b_i = 11k-3\}.$$

If d is not divisible by six, then $s = 0 = \#\{i \mid b_i = \frac{11}{6}d - 3\}$. □

4.5 Applications

Cuspidal plane sextics.

Theorem 4.13. *Let C be an irreducible cuspidal sextic with κ cusps and $\Delta_C(t) = (t^2-t+1)^s$. Then*

$$s = \begin{cases} 0 & \text{if } \kappa < 6, \\ \kappa - 6 & \text{if } \kappa > 6, \\ 1 & \text{if } \kappa = 6 \text{ and all cusps lie on a conic,} \\ 0 & \text{if } \kappa = 6 \text{ and the cusps do not lie on a conic.} \end{cases}$$

Proof:

- If C is smooth, then its Alexander polynomial is trivial by Theorem 1.1. Otherwise denote by I the ideal of cusps of C. Then $\kappa = \deg \operatorname{Proj} S/I$ and $s = \#\{i \mid b_i = 5\}$, where

$$0 \to \bigoplus_{i=1}^{t} S(-b_i) \to \bigoplus_{i=1}^{t+1} S(-a_i) \to S \to S/I \to 0$$

is a minimal graded free resolution of S/I in virtue of Corollary 4.8. Obviously $s \le t$. An upper bound for t may be obtained as follows: Suppose that all cusps of C lie on a curve of degree d. This is guaranteed if

$$\kappa \le \dim S_d - 1 = \binom{d+2}{2} - 1.$$

Solving this for d yields

$$d \ge \frac{1}{2}(\sqrt{8\kappa+9} - 3).$$

Now [11, Corollary 3.8] gives the upper bound

$$t \le \left\lceil \frac{1}{2}(\sqrt{8\kappa+9} - 3) \right\rceil =: T(\kappa).$$

- A sextic can have at most nine cusps due to Proposition 2.5. For $\kappa = 1, \dots, 9$ enumerate all sequences

$$(A_1, A_2, \dots, A_t, A_{t+1}, B_1, B_2 \dots, B_t) \in \mathbb{N}^{2t+1}, \qquad \text{where } t = 1, \dots, T(\kappa),$$

  satisfying

  (a) $B_i \ge A_i + 1 \ge A_{i+1} + 1$ for $i = 1, \dots, t$,

  (b) $A_1 \ge \dots \ge A_{t+1}$, $B_1 \ge \dots \ge B_t$,

  (c) $\sum_{i=1}^{t+1} A_i = \sum_{i=1}^{t} B_i$,

  (d) $\sum_{i=1}^{t} B_i^2 - \sum_{i=1}^{t+1} A_i^2 = 2\kappa$,

  (e) $B_1 \le 5$,

  (f) $A_{t+1} \ge \kappa/3$.

  In view of Section 4.4, all possible minimal graded free resolutions of S/I are enumerated. These are

κ	Resolution of S/I
1	$0 \to S(-2) \to S(-1)^2 \to S \to S/I \to 0$
2	$0 \to S(-3) \to S(-2) \oplus S(-1) \to \dots$
3	$0 \to S(-4) \to S(-3) \oplus S(-1) \to \dots$
	$0 \to S(-3)^2 \to S(-2)^3 \to \dots$
4	$0 \to S(-4) \to S(-2)^2 \to \dots$
	$0 \to S(-4) \oplus S(-3) \to S(-3) \oplus S(-2)^2 \to \dots$
5	$0 \to S(-4)^2 \to S(-3)^2 \oplus S(-2) \to \dots$
	$0 \to S(-5) \oplus S(-3) \to S(-4) \oplus S(-2)^2 \to \dots$
6	$0 \to S(-4)^3 \to S(-3)^4 \to \dots$
	$0 \to S(-5) \to S(-3) \oplus S(-2) \to \dots$
	$0 \to S(-5) \oplus S(-4) \to S(-4) \oplus S(-3) \oplus S(-2) \to \dots$
7	$0 \to S(-5) \oplus S(-4) \to S(-3)^3 \to \dots$
	$0 \to S(-5) \oplus S(-4)^2 \to S(-4) \oplus S(-3)^3 \to \dots$
8	$0 \to S(-5)^2 \to S(-4) \oplus S(-3)^2 \to \dots$
	$0 \to S(-5)^2 \oplus S(-4) \to S(-4)^2 \oplus S(-3)^2 \to \dots$
9	$0 \to S(-5)^3 \to S(-4)^3 \oplus S(-3) \to \dots$

Since $s = \#\{i \mid b_i = 5\}$, this yields that $s = 0$ if $\kappa < 5$ and $s = \kappa - 6$ if $\kappa > 6$. Moreover $s \in \{0, 1\}$ for $\kappa \in \{5, 6\}$. If $\kappa = 6$, then $s = \#\{i \mid b_i = 5\} = 1$ if and only if I has a generator of degree two, i. e. a conic.

- In order to prove the theorem, it suffices to show that

$$0 \to S(-5) \oplus S(-3) \to S(-4) \oplus S(-2)^2 \to S \to S/I \to 0$$

cannot be a minimal graded free resolution of S/I, where I is the ideal of cusps of a reduced five-cuspidal sextic C.

Assume the contrary. Then I is minimally generated by a quartic and two conics $q_1 \neq q_2$. Since q_1 and q_2 intersect in $5 > 4 = \deg q_1 \cdot \deg q_2$ points, they contain a common irreducible component by Bézout's theorem. Thus there are lines ℓ, ℓ_1, ℓ_2 such that $q_1 = \ell\ell_1$ and $q_2 = \ell\ell_2$. As $\ell_1 \neq \ell_2$ (otherwise $q_1 = q_2$), at least four cusps of C lie on the line ℓ. Again by Bézout, C must contain ℓ, that is, C is the union of ℓ and a quintic C'. But as a line cannot have cusps, C' has five cusps, four of which lie on ℓ. A final application of Bézout's theorem yields that ℓ is also a component of the quintic C'. This implies that C is not reduced, contradicting the assumption. $\square$

Thus the relationship between the number of cusps and the exponent of $t^2 - t + 1$ may be visualized in the following table:

κ	$s = 0$	$s = 1$	$s = 2$	$s = 3$
0	•			
1	•			
2	•			
3	•			
4	•			
5	•			
6	•	•		
7		•		
8			•	
9				•

Theorem 4.13 is sharp in the sense that a black dot at (κ, s) implies the existence of some cuspidal sextic curve with κ cusps and Alexander polynomial $(t^2 - t + 1)^s$. For details see Chapter 5.

The general case. In general, if C is a cuspidal curve of degree $6k$ for some $k \in \mathbb{N}$, then the connection between the number κ of cusps and the exponent s of $t^2 - t + 1$ in the Alexander polynomial is more sophisticated. An easy bound is:

Corollary 4.14. *With the above notation,*

$$s \geq \kappa - \binom{5k-1}{2}.$$

Proof: If Σ denotes the locus of cusps of C, then s is the dimension of the cokernel of

$$\varphi : S_{5k-3} \to \mathbb{C}^\kappa, \quad f \mapsto f(p)_{p \in \Sigma}$$

by Corollary 4.6. Now

$$\begin{aligned} s = \dim_{\mathbb{C}} \operatorname{coker} \varphi &= \dim_{\mathbb{C}} \mathbb{C}^\kappa - \dim_{\mathbb{C}} S_{5k-3} + \dim \ker \varphi \\ &\geq \dim_{\mathbb{C}} \mathbb{C}^\kappa - \dim_{\mathbb{C}} S_{5k-3} \\ &= \kappa - \binom{5k-1}{2}. \end{aligned}$$

□

Analyzing the possible minimal graded free resolutions of the ideal of cusps gives

Proposition 4.15 ([21]). *Suppose that $s \geq 1$. Then*

$$\kappa \geq \frac{3k}{2}\left(s - 1 + 2k + \sqrt{-s^2 + 4ks + 1 - 4k + 4k^2}\right).$$

Idea of proof: As in the proof of Theorem 4.13, the strategy is to consider all possible minimal graded free resolutions associated to the ideal of cusps I subject to the restrictions derived in the previous section. It turns out that it suffices to look at three special cases of sequences $(A_1, \dots, A_{t+1}, B_1, \dots, B_{t+1})$ where additionally $A_i \leq B_{i+1}$. Then the bound can be proved for each case individually.

Using the second bound on the number of cusps (Proposition 2.6), one obtains:

Corollary 4.16. *Again under the above assumptions,*

$$s \leq \frac{1}{4}\left(15k - 1 - \sqrt{15k^2 - 18k + 7}\right).$$

Proof: See [21, Theorem 5.2]. □

Consequences for degree 12. The relationship between the number of cusps of cuspidal curves of degree 12 and their Alexander polynomials can now be summarized as follows:

Theorem 4.17. *Let C be an irreducible cuspidal curve of degree 12 with κ cusps. The possible values for the exponent s of $t^2 - t + 1$ in $\Delta_C(t)$ are indicated with "$\circ$" or "$\bullet$" at position (κ, s) in the following table:*

κ	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$
0	$\bullet$					
$\vdots$	$\vdots$					
23	$\circ$					
24	$\bullet$	$\bullet$				
25	$\circ$	$\circ$				
26	$\bullet$	$\bullet$				
27	$\bullet$	$\bullet$				
28	$\bullet$	$\bullet$				
29	$\circ$	$\circ$	$\circ$			
30	$\bullet$	$\bullet$	$\circ$			
31	$\circ$	$\circ$	$\circ$			
32	$\bullet$	$\bullet$	$\bullet$			
33	$\circ$	$\circ$	$\bullet$	$\circ$		
34	$\circ$	$\bullet$	$\bullet$	$\circ$		
35	$\circ$	$\circ$	$\circ$	$\circ$		
36	$\circ$	$\circ$	$\circ$	$\bullet$	$\circ$	
37		$\circ$	$\circ$	$\circ$	$\circ$	
38			$\circ$	$\bullet$	$\circ$	
39				$\circ$	$\bullet$	$\circ$
40					$\circ$	$\circ$

Moreover, "$\bullet$" at (κ, s) means that there exists some cuspidal curve of degree 12 with κ cusps and $\Delta_C(t) = (t^2 - t + 1)^s$.

Proof: The first part is a straightforward consequence of Corollary 4.14 and Proposition 4.15. The dot at $(0, 0)$ follows from Theorem 1.1. The examples in Chapter 5 prove the rest of the "moreover" part. $\square$

5 Examples

Let $S := \mathbb{C}[x, y, z]$.

5.1 Strategy

Creating cuspidal curves of degree 12 using Kummer coverings. Suppose that C is a cuspidal curve of degree d. Consider the Kummer covering φ_n. As remarked in Chapter 2, if C has a sufficiently nice position, then $\varphi_n^{-1}(C)$ is also a cuspidal curve and has degree nd. This will be clarified here.

For example, let C be a cuspidal quartic. Assume that C satisfies the prerequisites of Proposition 2.8 and that C intersects $V(xyz)$ with multiplicity at most two at each point in $C \cap V(xyz)$. Then applying the Kummer covering φ_3 yields a cuspidal curve of degree 12. This motivates the following recipe, called the Kummer covering ramified along ℓ_1, ℓ_2, ℓ_3:

(1) Given a cuspidal quartic C, pick three linearly independent lines ℓ_1, ℓ_2, ℓ_3 such that each of these lines either intersects C transversely or is an ordinary (bi)tangent to C.

(2) Perform a coordinate change such that ℓ_1, ℓ_2, ℓ_3 become the lines $V(x), V(y), V(z)$, respectively.

(3) If C does not contain any of the points $(1:0:0), (0:1:0), (0:0:1)$, then $\varphi_3^{-1}(C)$ is a cuspidal curve of degree 12.

Suppose that C has κ cusps and that there are m points of C where the curve intersects $V(xyz)$ with multiplicity two. Then $\varphi_3^{-1}(C)$ has $9\kappa + 3m$ cusps.

An analogous method may be applied for sextics: One requires here that ℓ_1, ℓ_2, ℓ_3 either intersect C transversely or are inflectional (bi)tangents to C. Applying the Kummer covering φ_2 yields a cuspidal curve of degree 12. If the base sextic has κ cusps and there are m points of C where the sextic intersects $V(xyz)$ with multiplicity three, then $\varphi_2^{-1}(C)$ has $4\kappa + 2m$ cusps.

Obtaining cuspidal curves of degree 12 by applying φ_4 to cubics, φ_6 to conics or even φ_{12} to lines is not very helpful: In all these cases, the pullback via the corresponding morphism φ_n is only a cuspidal curve if $V(xyz)$ intersects the base curve transversely. Therefore the number of cusps of the pullback will be equal to n^2 times the number of cusps of the base curve. Since lines and conics cannot have cusps, and cubics can have at most one cusp, the pullback curve will have at most 16 cusps and the Alexander polynomial will always be trivial.

Behavior of the Alexander polynomial.

Corollary 5.1. *Let $C = V(f)$ be an irreducible cuspidal curve satisfying the prerequisites of Proposition 2.8. Let further $n \in \mathbb{N}$ and assume that $\varphi_n^{-1}(C)$ is also an irreducible cuspidal curve. Then*

(a) $\Delta_C(t) \mid \Delta_{\varphi_n^{-1}(C)}(t)$.

(b) If C intersects $V(xyz)$ transversely, i. e. $\Delta = \emptyset$ in the notation of Proposition 2.8, then the exponents of the factor $t^2 - t + 1$ in $\Delta_C(t)$ and $\Delta_{\varphi_n^{-1}(C)}(t)$ coincide.

<u>Proof:</u> By Theorem 3.6,

$$\Delta_C(t) = (t^2 - t + 1)^s, \quad \Delta_{\varphi_n^{-1}(C)}(t) = (t^2 - t + 1)^{s'}$$

for some $s, s' \in \mathbb{N}_0$.

Let I be the ideal of cusps of C and I_n the ideal of cusps of $\varphi_n^{-1}(C)$. By Proposition 2.8 and Lemma 4.7, $\varphi_n^*(I)$ contains I_n. Now, if C has degree $6k$, then $\varphi_n^{-1}(C)$ has degree $6kn$. Due to Corollary 4.8,

$$h_{S/\varphi_n^*(I)}(5kn - 3) = h_{S/I}(5k - 3) = s \quad \text{and} \quad h_{S/I_n}(5kn - 3) = s'.$$

Combining this with

$$h_{S/I_n}(5kn - 3) - h_{S/\varphi_n^*(I)}(5kn - 3) = h_{\varphi_n^*(I)/I_n}(5kn - 3) \geq 0$$

yields that $s' \geq s$. This shows (a).

For (b) note that if $\Delta = \emptyset$, then Proposition 2.8 and Lemma 4.7 imply $\varphi_n^*(I) = I_n$. □

The covering φ_n will be called general if the assumptions of (b) are satisfied, which can be assumed after a change of coordinates.

Computation of the Alexander polynomial. If C' is a curve obtained by a Kummer covering of some irreducible cuspidal curve C ramified along three lines, check at first whether C' is irreducible. If the covering was general, then by Corollary 5.1 (b) the Alexander polynomials of C and C' coincide. Otherwise, compute a minimal graded free resolution of one of the three ideals of cusps of Chapter 4. This will be done using the computer algebra system Singular ([5]). In the most cases, the Jacobian ideal is the easiest for this purpose. Finally apply Corollary 4.8, Corollary 4.9 or Proposition 4.12, i. e. count the number of syzygies of degrees 10, 14 or 19, respectively.

Index of the examples. The following table is basically the same as in Theorem 4.17. An example for a curve with κ cusps and Alexander polynomial $(t^2 - t + 1)^s$ can be found in the next two sections at the indicated label.

κ	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$
$\vdots$	$\vdots$					
23	$\circ$					
24	(4.1), (6.5)	(4.2), (6.1)				
25	$\circ$	$\circ$				
26	(6.6)	(6.2)				
27	(4.3), (4.6)	(4.4)				
28	(6.7), (6.8)	(6.3), (6.14)				
29	$\circ$	$\circ$	$\circ$			
30	(6.9), (6.10)	(4.5), (4.7), (6.4), (6.15)	$\circ$			
31	$\circ$	$\circ$	$\circ$			
32	(6.11)	(6.12), (6.16)	(6.18), (6.22)			
33	$\circ$	$\circ$	(4.8), (4.9)	$\circ$		
34	$\circ$	(6.13), (6.17)	(6.19), (6.23)	$\circ$		
35	$\circ$	$\circ$	$\circ$	$\circ$		
36	$\circ$	$\circ$	$\circ$	(4.10), (4.11), (6.20) (6.24), (6.25)	$\circ$	
37		$\circ$	$\circ$	$\circ$	$\circ$	
38			$\circ$	(6.21)	$\circ$	
39				$\circ$	(4.12)	$\circ$
40					$\circ$	$\circ$

5.2 Coverings of quartic curves

Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a cuspidal quartic. If one requires the Kummer covering along three lines to produce a curve with at least 24 cusps, then C needs to have at least two cusps. By Proposition 2.5, C can have at most three cusps.

Quartic with two cusps

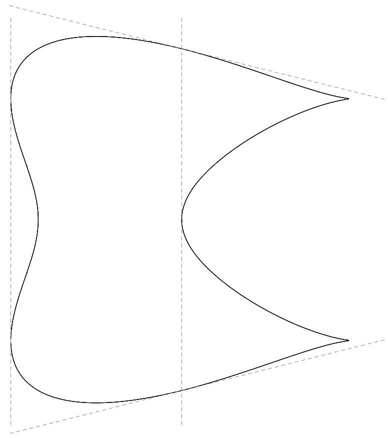
Consider the curve C defined via

$$(x^2 - z^2)(x - z)^2 + (2y^2 - z^2)^2 = 0.$$

C is a two-cuspidal quartic with cusps at $(1 : 1 : 1)$ and $(1 : -1 : 1)$. Define the following tangent lines:

$$b := V(x + z), \quad t_1 := V(x), \quad t_2 := V(x + 4y - 4z), \quad t_3 := V(x - 4y - 4z).$$

The line b is the single bitangent to C and t_1, t_2, t_3 are the tangents at $(0 : 0 : 1)$, $(0 : 1 : 1)$, $(0 : -1 : 1)$, respectively. Further pick the lines $\ell_1 := V(2y - z)$ and $\ell_2 := V(2y + z)$, that intersect C transversely.



The two-cuspidal quartic $V\left((x^2-1)(x-1)^2+(2y^2-1)^2\right) \subset \mathbb{R}^2$

(4.1) Covering ramified along t_2, t_3 and ℓ_1. This produces the curve $C_{4.1}$ defined by the irreducible polynomial

$$3501x^{12} - 3604x^9y^3 + 1326x^6y^6 - 212x^3y^9 + 13y^{12} - 35232x^9z^3 + 26976x^6y^3z^3 \\ - 6624x^3y^6z^3 + 544y^9z^3 + 132800x^6z^6 - 67456x^3y^3z^6 + 8384y^6z^6 - 222208x^3z^9 \\ + 56320y^3z^9 + 139264z^{12}.$$

$C_{4.1}$ is a cuspidal curve of degree 12 with 24 cusps. Let J be the Jacobian ideal. Then S/J has the following minimal graded free resolution:

$$0 \to S(-27) \oplus S(-24) \oplus S(-23) \to S(-22)^3 \oplus S(-21) \oplus S(-20) \to S(-11)^3 \to S \to S/J \to 0$$

By Proposition 4.12, this implies that $\Delta_{C_{4.1}}(t) = 1$.

(4.2) Covering ramified along b, ℓ_1 and ℓ_2. The result is the curve $C_{4.2}$

$$64x^{12} + 192x^9y^3 + 192x^6y^6 + 64x^3y^9 + y^{12} - 192x^9z^3 - 384x^6y^3z^3 - 192x^3y^6z^3 - 12y^9z^3 \\ + 192x^6z^6 + 192x^3y^3z^6 + 38y^6z^6 - 64x^3z^9 - 12y^3z^9 + z^{12} = 0.$$

This is an irreducible curve with 24 cusps as the only singularities. If J denotes the Jacobian ideal, then S/J has the minimal resolution

$$0 \to S(-27) \oplus S(-25) \to S(-22)^3 \oplus S(-19) \to S(-11)^3 \to S \to S/J \to 0.$$

Hence $\Delta_{C_{4.2}}(t) = t^2 - t + 1$.

(4.3) Covering ramified along t_1, t_2 and t_3. For this covering, consider the following slight modification of the base curve C:

$$(x^2 - z^2)(x-z)^2 + ((1+a)y^2 - az^2)^2 = 0, \qquad \text{where } a := \sqrt{3}/4.$$

The tangents t_2 and t_3 become $V(2x \pm (4+\sqrt{3})y - (4+\sqrt{3})z)$, and t_1 becomes $V(2x-z)$. The Kummer covering φ_3 ramified along these new tangent lines produce the irreducible 27-cuspidal curve $C_{4.3}$ given by the vanishing of

$$\begin{aligned}&243x^{12}+129\sqrt{3}x^{12}-8244x^9y^3-4752\sqrt{3}x^9y^3+16002x^6y^6+9246\sqrt{3}x^6y^6-8244x^3y^9\\&-4752\sqrt{3}x^3y^9+243y^{12}+129\sqrt{3}y^{12}-3012x^9z^3-1624\sqrt{3}x^9z^3-38124x^6y^3z^3\\&-21816\sqrt{3}x^6y^3z^3-38124x^3y^6z^3-21816\sqrt{3}x^3y^6z^3-3012y^9z^3-1624\sqrt{3}y^9z^3\\&-18216x^6z^6-10872\sqrt{3}x^6z^6-7344x^3y^3z^6-4800\sqrt{3}x^3y^3z^6-18216y^6z^6-10872\sqrt{3}y^6z^6\\&+167184x^3z^9+96864\sqrt{3}x^3z^9+167184y^3z^9+96864\sqrt{3}y^3z^9+874416z^{12}+504656\sqrt{3}z^{12}.\end{aligned}$$

The minimal resolution of the Jacobian ideal of $C_{4.3}$ is

$$\begin{aligned}&0\to S(-25)\oplus S(-24)^2\oplus S(-23)\to S(-22)^4\oplus S(-21)\oplus S(-20)\\&\quad\to S(-11)^3\to S\to S/J\to 0,\end{aligned}$$

thus $\Delta_{C_{4.3}}(t)=1$.

(4.4) Covering ramified along b, t_1 and ℓ_1. This results in the curve $C_{4.4}$ given by

$$\begin{aligned}&3x^{12}-20x^9y^3+42x^6y^6-28x^3y^9-y^{12}+4x^9z^3-12x^6y^3z^3+12x^3y^6z^3-4y^9z^3-2x^6z^6\\&+4x^3y^3z^6-2y^6z^6-4x^3z^9+4y^3z^9-z^{12}=0.\end{aligned}$$

It has 27 cusps and Alexander polynomial t^2-t+1: The Jacobian ideal J has the free resolution

$$0\to S(-25)^2\oplus S(-24)\to S(-22)^4\oplus S(-19)\to S(-11)^3\to S\to S/J\to 0.$$

(4.5) Covering ramified along b, t_2 and t_3. This produces the curve $C_{4.5}$ with the equation

$$\begin{aligned}&139264x^{12}+136192x^9y^3+46016x^6y^6+6208x^3y^9+289y^{12}+136192x^9z^3+98432x^6y^3z^3\\&+12224x^3y^6z^3-2244y^9z^3+46016x^6z^6+12224x^3y^3z^6+4934y^6z^6+6208x^3z^9-2244y^3z^9\\&+289z^{12}=0.\end{aligned}$$

This is a 30-cuspidal curve with Alexander polynomial t^2-t+1. A minimal resolution of the Jacobian ideal is given by

$$0\to S(-25)\oplus S(-24)\oplus S(-23)\to S(-22)^3\oplus S(-20)\oplus S(-19)\to S(-11)^3\to S\to S/J\to 0.$$

Quartic with three cusps

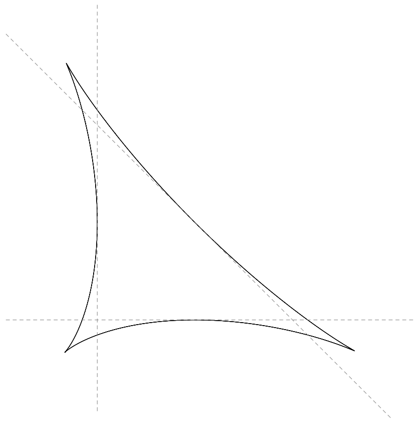
Define C to be the vanishing locus in $\mathbb{P}^2_{\mathbb{C}}$ of the homogeneous polynomial

$$x^2y^2+x^2z^2+y^2z^2-2xyz(x+y+z).$$

Then C is a three-cuspidal quartic whose cusps are situated in $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$. Its single bitangent is given by $b := V(x+y+z)$. Define the following tangent lines:

$$
\begin{aligned}
t_1 &:= V(-8x+y+z) && \text{at } (1:4:4),\\
t_2 &:= V(x-8y+z) && \text{at } (4:1:4),\\
t_3 &:= V(x+y-8z) && \text{at } (4:4:1).
\end{aligned}
$$

Further choose the two general lines $\ell_1 := V(x+y-z)$ and $\ell_2 := V(x-y-z)$.



Real part of C and t_1, t_2, t_3 in the affine chart $x+y+z=1$

(4.6) General covering. Due to Theorem 3.6 and Corollary 5.1, this produces a cuspidal curve of degree 12 with 27 cusps and Alexander polynomial 1.

(4.7) Covering ramified along t_1, ℓ_1 and ℓ_2. An equation for the pullback curve under the covering is given by

$$
\begin{aligned}
16x^{12} &+ 336x^9y^3 + 1372x^6y^6 + 1372x^3y^9 + 2401y^{12} - 48x^9z^3 + 1400x^6y^3z^3 + 4116x^3y^6z^3\\
&+ 1372y^9z^3 - 1220x^6z^6 + 420x^3y^3z^6 - 2058y^6z^6 - 3604x^3z^9 - 644y^3z^9 - 47z^{12} = 0.
\end{aligned}
$$

This curve has 30 cusps. Computing a minimal graded free resolution of S/J, where J is the Jacobian ideal, shows that the Alexander polynomial equals $t^2 - t + 1$.

(4.8) Covering ramified along t_1, t_2 and ℓ_1. The result is the 33-cuspidal curve $C_{4.8}$ defined via

$$
\begin{aligned}
47x^{12} &+ 11638x^9y^3 + 22557x^6y^6 + 11638x^3y^9 + 47y^{12} + 2916x^9z^3 + 154548x^6y^3z^3\\
&+ 154548x^3y^6z^3 + 2916y^9z^3 + 54918x^6z^6 + 559386x^3y^3z^6 + 54918y^6z^6\\
&+ 288684x^3z^9 + 288684y^3z^9 + 177147z^{12} = 0.
\end{aligned}
$$

A minimal resolution of the Jacobian ideal J is

$$0 \to S(-24)^2 \to S(-22) \oplus S(-21) \oplus S(-19)^2 \to S(-11)^3 \to S/J \to S \to 0.$$

Hence $C_{4.8}$ has the Alexander polynomial $(t^2 - t + 1)^2$.

(4.9) Covering ramified along b, ℓ_1, ℓ_2. This yields also a curve $C_{4.9}$ of degree 12 with 33 cusps. Its equation is

$$\begin{aligned} &x^{12} - 6x^9y^3 + 7x^6y^6 - 2x^3y^9 + y^{12} + 6x^9z^3 - 16x^6y^3z^3 + 12x^3y^6z^3 - 2y^9z^3 + 7x^6z^6 \\ &- 12x^3y^3z^6 + 3y^6z^6 + 2x^3z^9 - 2y^3z^9 + z^{12} = 0, \end{aligned}$$

the minimal resolution of the Jacobian ideal J is given by

$$0 \to S(-24)^2 \oplus S(-22) \to S(-22)^2 \oplus S(-21) \oplus S(-19)^2 \to S(-11)^3 \to S/J \to S \to 0.$$

Again, the Alexander polynomial equals $(t^2 - t + 1)^2$.

(4.10) Covering ramified along t_1, t_2, t_3. The covering gives the curve $C_{4.10}$ with 36 cusps with defining polynomial

$$\begin{aligned} &x^{12} + 12x^9y^3 - 26x^6y^6 + 12x^3y^9 + y^{12} + 12x^9z^3 + 244x^6y^3z^3 + 244x^3y^6z^3 + 12y^9z^3 \\ &- 26x^6z^6 + 244x^3y^3z^6 - 26y^6z^6 + 12x^3z^9 + 12y^3z^9 + z^{12}. \end{aligned}$$

Minimal resolution of the Jacobian ideal J:

$$0 \to S(-24) \to S(-19)^3 \to S(-11)^3 \to S/J \to S \to 0.$$

Hence $\Delta_{C_{4.10}}(t) = (t^2 - t + 1)^3$.

(4.11) Covering ramified along b, t_1 and ℓ_1. This does also produce a degree 12 curve with 36 cusps. Its equation is

$$\begin{aligned} &47x^{12} - 4580x^9y^3 - 1932x^6y^6 - 80x^3y^9 - 16y^{12} - 828x^9z^3 - 1404x^6y^3z^3 + 2376x^3y^6z^3 \\ &- 144y^9z^3 + 3402x^6z^6 + 10692x^3y^3z^6 - 972y^6z^6 + 2916x^3z^9 - 2916y^3z^9 - 6561z^{12} = 0. \end{aligned}$$

The minimal resolution of the Jacobian ideal J is

$$0 \to S(-24) \oplus S(-22) \to S(-22) \oplus S(-19)^3 \to S(-11)^3 \to S/J \to S \to 0.$$

The Alexander polynomial is again $(t^2 - t + 1)^3$.

(4.12) Covering ramified along b, t_1 and t_2 ([24, Example 6.3]). The resulting curve $C_{4.12}$ has the equation

$$\begin{aligned} &27x^{12} - 36x^9y^3 - 42x^6y^6 - 12x^3y^9 - y^{12} - 36x^9z^3 + 138x^6y^3z^3 + 12x^3y^6z^3 - 2y^9z^3 \\ &- 42x^6z^6 + 12x^3y^3z^6 - 3y^6z^6 - 12x^3z^9 - 2y^3z^9 - z^{12} = 0. \end{aligned}$$

The minimal graded free resolution of the Jacobian ideal J is

$$0 \to S(-22) \oplus S(-21) \to S(-19)^4 \to S(-11)^3 \to S/J \to S \to 0.$$

Thus the Alexander polynomial equals $(t^2 - t + 1)^4$. Note that $C_{4.12}$ has 39 cusps, which shows that the bound on the number of cusps given in Proposition 2.5 is quite realistic.

5.3 Coverings of sextic curves

In this section, Kummer coverings of six-, seven-, eight- and nine-cuspidal sextics are computed. In order to construct such curves, the so-called sextics of torus type are useful: Suppose that $C_2 = V(f_2)$ is a conic and that $C_3 = V(f_3)$ is a cubic in $\mathbb{P}^2_{\mathbb{C}}$. The intersection $C_2 \cap C_3$ consists of six points counted with multiplicity by Bézout's theorem. If these six points are all distinct, then the curve $C := V(f_2^3 + f_3^2)$ has six cusps at precisely these points. A curve arising in this way is called of torus type. In fact, one can show:

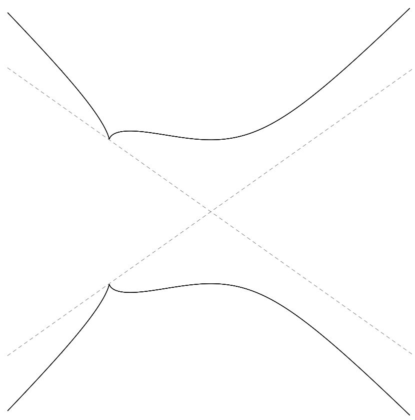
Theorem 5.2. *Let C be an irreducible cuspidal sextic. Then C is of torus type if and only if $\Delta_C(t)$ is not trivial.*

Proof: If $C = V(f)$ is of torus type, then f has a quasi-toric relation of type $(2,3,6)$. Hence Proposition 2.13 implies that $\operatorname{rk} \mathrm{MW}(X_f) > 0$. By Theorem 3.9, this means that $\deg \Delta_C(t) > 0$. For the converse see [7, Theorem 1.1.2] or [28, Theorem 0.4]. □

In particular, all presented sextics in the sequel (with the exception of the non-conical six-cuspidal sextic) are of torus type.

In the case of Kummer coverings of sextics, one looks for inflectional (bi)tangents. These are harder to find than (bi)tangents in the quartic case. Moreover, they are often not defined over $\mathbb{Q}$. Thus sometimes, it is infeasible to compute a minimal graded free resolution or even the Hilbert function of one of the three ideals of cusps. If this happens, then reductions modulo a prime number are considered, which give an upper bound on the degree of the Alexander polynomial. This will be explained in Example (6.4).

Sextic with six cusps on a conic



$V\left((x^2-2y^2)^3 + (x^3+1)^2\right)$ and the degenerate conic $V(x^2-2y^2)$ in $\mathbb{R}^2$

Consider the curve C given by

$$(x^2 - 2y^2)^3 + (x^3 + z^3)^2 = 0.$$

This is an irreducible sextic with six cusps as the only singularities. All the cusps are situated on the six intersection points of the conic $V(x^2 - 2y^2)$ with the cubic $V(x^3 + z^3)$. Hence by Theorem 4.13, $\Delta_C(t) = t^2 - t + 1$. Pick the two inflectional tangents $t_{1/2} := V(x \pm y)$ at $(1 : \mp 1 : 0)$. Further choose the lines $\ell_1 := V(z)$ and $\ell_2 := V(x + 2y + z)$ that intersect C transversely.

(6.1) General covering. This produces a curve with 24 cusps and Alexander polynomial $t^2 - t + 1$ following Corollary 5.1 (b).

(6.2) Covering ramified along t_1, ℓ_1 and ℓ_2. The result is the curve $C_{6.2}$ defined via

$$\begin{aligned}
&72x^{12} + 192x^{10}y^2 + 228x^8y^4 + 176x^6y^6 + 90x^4y^8 + 24x^2y^{10} + 3y^{12} - 192x^{10}z^2 - 456x^8y^2z^2 \\
&- 480x^6y^4z^2 - 288x^4y^6z^2 - 84x^2y^8z^2 - 6y^{10}z^2 + 228x^8z^4 + 480x^6y^2z^4 + 396x^4y^4z^4 \\
&+ 132x^2y^6z^4 + 6y^8z^4 - 160x^6z^6 - 264x^4y^2z^6 - 120x^2y^4z^6 - 2y^6z^6 + 66x^4z^8 + 60x^2y^2z^8 \\
&- 12x^2z^{10} = 0.
\end{aligned}$$

This curve has 26 cusps and Alexander polynomial $t^2 - t + 1$, as a minimal graded free resolution of S/J, where J is the Jacobian ideal, is given by

$$0 \to S(-26) \oplus S(-25) \to S(-22)^2 \oplus S(-21) \oplus S(-19) \to S(-11)^3 \to S \to S/J \to 0.$$

(6.3) Covering ramified along t_1, t_2 and ℓ_2 This gives

$$\begin{aligned}
&675x^{12} - 1542x^{10}y^2 + 1101x^8y^4 - 212x^6y^6 + 45x^4y^8 - 6x^2y^{10} + 3y^{12} - 2808x^{10}z^2 \\
&+ 5112x^8y^2z^2 - 3120x^6y^4z^2 + 1008x^4y^6z^2 - 216x^2y^8z^2 + 24y^{10}z^2 + 4788x^8z^4 - 6672x^6y^2z^4 \\
&+ 3096x^4y^4z^4 - 720x^2y^6z^4 + 84y^8z^4 - 4304x^6z^6 + 4368x^4y^2z^6 - 1392x^2y^4z^6 + 176y^6z^6 \\
&+ 2160x^4z^8 - 1440x^2y^2z^8 + 240y^4z^8 - 576x^2z^{10} + 192y^2z^{10} + 64z^{12} = 0.
\end{aligned}$$

The minimal resolution of the Jacobian ideal is

$$0 \to S(-25)^2 \oplus S(-23) \to S(-22)^3 \oplus S(-21) \oplus S(-19) \to S(-11)^3 \to S \to S/J \to 0.$$

This is an example of a cuspidal curve of degree 12 with 28 cusps and Alexander polynomial $t^2 - t + 1$.

(6.4) Covering ramified along three inflectional tangents. One finds that a third inflectional tangent of C is given by $V(x + ay + bz)$, where $a^6 - 39a^4 - 10a^2 - 4 = 0$ and $b^9 + 78b^6 - 40b^3 + 32 = 0$. Computing a minimal free resolution of any of the ideals of cusps of the curve $C_{6.4}$ turns out to be infeasible. Thus work in positive characteristic. The smallest prime p where the minimal polynomials for a and b split and the reduction $\overline{C_{6.4}}^p$ modulo p of $C_{6.4}$ has still 30 cusps and no further singularities, is $p = 66751$. An equation for $\overline{C_{6.4}}^{66751}$ is given by

$$\begin{aligned}
&-25230x^{12}-3738x^{10}y^2-12543x^8y^4-15763x^6y^6+17200x^4y^8+29861x^2y^{10}+18489y^{12}\\
&+31730x^{10}z^2-13101x^8y^2z^2-29625x^6y^4z^2-191x^4y^6z^2+28878x^2y^8z^2-9627y^{10}z^2\\
&-7674x^8z^4-4185x^6y^2z^4-252x^4y^4z^4+25917x^2y^6z^4+12974y^8z^4+31631x^6z^6\\
&-27967x^4y^2z^6-16899x^2y^4z^6+12986y^6z^6+29725x^4z^8-18156x^2y^2z^8+24254y^4z^8\\
&-21609x^2z^{10}+20630y^2z^{10}-10962z^{12}=0.
\end{aligned}$$

The theory of free resolutions and Hilbert functions as presented in Section 4.1 can easily be adapted for the graded ring $\overline{S} := \mathbb{F}_{66571}[x,y,z]$. A minimal graded free resolution of the ideal $\overline{I}$ of cusps of $\overline{C_{6.4}}^{66751}$ is given by

$$0 \to \overline{S}(-10) \oplus \overline{S}(-8)^3 \to \overline{S}(-7)^4 \oplus \overline{S}(-6) \to \overline{S} \to \overline{S}/\overline{I} \to 0.$$

Computing the Hilbert function of $\overline{S}/\overline{I}$ from this resolution, one sees $\dim \overline{I}_7 = 7$. Following [3, Proposition 2.5], $\dim_{\mathbb{C}} I_7 \leq \dim_{\mathbb{C}} \overline{I}_7$, where I is the ideal of cusps of $C_{6.4}$. Consequently, if s denotes the exponent of t^2-t+1 in $\Delta_{C_{6.4}}(t)$, then

$$s = p_{S/I}(7) - h_{S/I}(7) = p_{S/I}(7) - \dim_{\mathbb{C}} S_7 + \dim_{\mathbb{C}} I_7 \leq 30 - 36 + 7 = 1.$$

By Corollary 5.1 (a), $s \geq 1$. Therefore $\Delta_{C_{6.4}}(t) = t^2 - t + 1$.

Sextic with six cusps not on a conic

Let ξ be a primitive third root of unity. Then the curve C given by the vanishing of

$$(y^2+\xi x^2)^3 - (x^2+\xi y^2)^3 - (z^2+\xi y^2+\xi^2 x^2)^3$$

defines an irreducible six-cuspidal sextic whose cusps do not lie on a conic. In particular, $\Delta_C(t) = 1$ in virtue of Theorem 4.13 or Theorem 5.2. The curve C can be obtained via a Kummer covering of degree 2 of a smooth cubic ramified along three special inflectional tangents (see [3, Section 1]).

This curve is a lucky curve: It has two inflectional bitangents $b_{1/2}$ given by $V(x \pm y)$. Pick further the lines $\ell_{1/2} := V(x \pm y + z)$ that intersect C transversely.

C is also an unlucky curve: All other inflectional tangents are not defined over $\mathbb{Q}(\xi)$. Thus we will have to work in positive characteristic as in (6.4).

(6.5) General covering. This produces a cuspidal degree 12 curve with 24 cusps and trivial Alexander polynomial.

(6.6) Covering ramified along one inflectional tangent, ℓ_1 and ℓ_2. The minimal field of definition of the resulting curve is computationally infeasible. Thus use the same methods as in (6.4): The reduction $\overline{C_{6.6}}^{229}$ of $C_{6.6}$ modulo 229 has 26 cusps and no further singularities. An equation is

$$\begin{aligned}
&-57x^{12} - 64x^{10}y^2 + 10x^8y^4 + 114x^6y^6 - 100x^4y^8 + 71x^2y^{10} + 32y^{12} + 95x^{10}z^2 + 13x^8y^2z^2 \\
&+ 32x^6y^4z^2 - 4x^4y^6z^2 + 79x^2y^8z^2 + 56y^{10}z^2 - 19x^8z^4 - 88x^6y^2z^4 + 12x^4y^4z^4 - 21x^2y^6z^4 \\
&- 54y^8z^4 - 44x^6z^6 - 69x^4y^2z^6 - 100x^2y^4z^6 - 26y^6z^6 + 15x^4z^8 - 47x^2y^2z^8 - 16y^4z^8 \\
&- 74x^2z^{10} - 31y^2z^{10} + 55z^{12} = 0.
\end{aligned}$$

The ideal $\overline{I} \trianglelefteq \overline{S} := \mathbb{F}_{229}[x, y, z]$ of cusps of $\overline{C_{6.6}}^{229}$ has the minimal resolution

$$0 \to \overline{S}(-9) \oplus \overline{S}(-8)^2 \to \overline{S}(-7) \oplus \overline{S}(-6)^3 \to \overline{S} \to \overline{S}/\overline{I} \to 0.$$

Analogously to (6.4), one obtains that $C_{6.6}$ is a 26-cuspidal curve with Alexander polynomial $\Delta_{C_{6.6}}(t) = 1$.

(6.7) Covering ramified along two inflectional tangents and ℓ_1. Stay in characteristic 229. The reduction $\overline{C_{6.7}}^{229}$ of $C_{6.7}$ has 28 cusps and is given by

$$\begin{aligned}
&30x^{12} - 108x^{10}y^2 - 114x^8y^4 + 64x^6y^6 + 54x^4y^8 + 37x^2y^{10} - 20y^{12} + 87x^{10}z^2 + 29x^8y^2z^2 \\
&- 13x^6y^4z^2 - 3x^4y^6z^2 - 98x^2y^8z^2 + 29y^{10}z^2 - 62x^8z^4 - 27x^6y^2z^4 - 45x^4y^4z^4 - 42x^2y^6z^4 \\
&- 49y^8z^4 - 32x^6z^6 + 103x^4y^2z^6 - 56x^2y^4z^6 - y^6z^6 + 72x^4z^8 + 114x^2y^2z^8 - 103y^4z^8 \\
&+ 27x^2z^{10} + 110y^2z^{10} + 16z^{12} = 0.
\end{aligned}$$

The ideal $\overline{I} \trianglelefteq \overline{S} := \mathbb{F}_{229}[x, y, z]$ of cusps of $\overline{C_{6.7}}^{229}$ has the minimal resolution

$$0 \to \overline{S}(-9)^2 \oplus \overline{S}(-8) \to \overline{S}(-7)^2 \oplus \overline{S}(-6)^2 \to \overline{S} \to \overline{S}/\overline{I} \to 0.$$

Consequently, $\Delta_{C_{6.7}}(t) = 1$.

(6.8) Covering ramified along b_1, ℓ_1, ℓ_2. Here, one can switch back to characteristic zero. This covering produces the curve $C_{6.8}$ with the defining polynomial

$$\begin{aligned}
&64x^{12} + (-192\xi - 384)x^{10}y^2 + (624\xi + 672)x^8y^4 + (-768\xi - 384)x^6y^6 + (420\xi - 60)x^4y^8 \\
&+ (-84\xi + 120)x^2y^{10} - 27y^{12} + (-192\xi)x^{10}z^2 + (672\xi - 192)x^8y^2z^2 \\
&+ (-960\xi + 384)x^6y^4z^2 + (624\xi - 336)x^4y^6z^2 + (-156\xi + 192)x^2y^8z^2 - 54y^{10}z^2 \\
&+ (-336\xi - 288)x^8z^4 + (1152\xi + 768)x^6y^2z^4 + (-1224\xi - 744)x^4y^4z^4 \\
&+ (408\xi + 240)x^2y^6z^4 - 9y^8z^4 + (-192\xi - 256)x^6z^6 + (432\xi + 432)x^4y^2z^6 \\
&+ (-216\xi - 192)x^2y^4z^6 + 28y^6z^6 + (-60\xi - 60)x^4z^8 + (60\xi + 24)x^2y^2z^8 + 3y^4z^8 \\
&+ (-12\xi)x^2z^{10} - 6y^2z^{10} + z^{12}.
\end{aligned}$$

The Jacobian ideal J has the following minimal resolution:

$$0 \to S(-26) \oplus S(-23)^2 \to S(-22)^2 \oplus S(-21) \oplus S(-20)^2 \to S(-11)^3 \to S \to S/J \to 0$$

Hence $C_{6.8}$ is an example of a degree 12 curve with 28 cusps and Alexander polynomial 1.

(6.9) Covering ramified along three inflectional tangents. Return to $\mathbb{F}_{229}$. The reduction $\overline{C_{6.9}}^{229}$ of $C_{6.9}$ has 30 cusps and is found to be

$$\begin{aligned}&8x^{12}-54x^{10}y^2+19x^8y^4+59x^6y^6+57x^4y^8-113x^2y^{10}+13y^{12}-6x^{10}z^2+57x^8y^2z^2\\&-32x^6y^4z^2+46x^4y^6z^2+30x^2y^8z^2+35y^{10}z^2+98x^8z^4-27x^6y^2z^4+92x^4y^4z^4+49x^2y^6z^4\\&-84y^8z^4-16x^6z^6+51x^4y^2z^6+32x^2y^4z^6-104y^6z^6-72x^4z^8-90x^2y^2z^8+61y^4z^8\\&+32x^2z^{10}+98y^2z^{10}-11z^{12}=0.\end{aligned}$$

The ideal $\overline{I}$ of cusps has the minimal resolution

$$0\to\overline{S}(-9)^3\to\overline{S}(-7)^3\oplus\overline{S}(-6)\to\overline{S}\to\overline{S}/\overline{I}\to 0.$$

The Alexander polynomial is hence $\Delta_{C_{6.9}}(t)=1$.

(6.10) Covering ramified along b_1, one inflectional tangent and ℓ_1. The reduction $\overline{C_{6.10}}^{229}$ of $C_{6.10}$ is a curve with 30 cusps as the only singularities. A defining polynomial in $\mathbb{F}_{229}[x,y,z]$ is given by

$$\begin{aligned}&36x^{12}-100x^{10}y^2+2x^8y^4+71x^6y^6+36x^4y^8+19x^2y^{10}-43y^{12}-2x^{10}z^2-50x^8y^2z^2\\&+80x^6y^4z^2-49x^4y^6z^2-84x^2y^8z^2+30y^{10}z^2+92x^8z^4-68x^6y^2z^4+109x^4y^4z^4\\&-48x^2y^6z^4+89y^8z^4+61x^6z^6-50x^4y^2z^6+50x^2y^4z^6+26y^6z^6-77x^4z^8-62x^2y^2z^8\\&-112y^4z^8+22x^2z^{10}+65y^2z^{10}-44z^{12}.\end{aligned}$$

The ideal $\overline{I}$ of cusps has the minimal resolution

$$0\to\overline{S}(-9)^3\to\overline{S}(-7)^3\oplus\overline{S}(-6)\to\overline{S}\to\overline{S}/\overline{I}\to 0.$$

As a consequence, again $\Delta_{C_{6.10}}(t)=1$.

(6.11) Covering ramified along b_1 and two inflectional tangents. The reduction $\overline{C_{6.11}}^{229}$ of $C_{6.11}$ is a curve with 32 cusps as the only singularities defined via

$$\begin{aligned}&-17x^{12}-76x^{10}y^2+52x^8y^4-89x^6y^6-88x^4y^8+109x^2y^{10}+17y^{12}-64x^{10}z^2+3x^8y^2z^2\\&+21x^6y^4z^2+106x^4y^6z^2-45x^2y^8z^2+81y^{10}z^2-55x^8z^4-31x^6y^2z^4-45x^4y^4z^4\\&-104x^2y^6z^4-95y^8z^4+81x^6z^6+40x^4y^2z^6-93x^2y^4z^6+55y^6z^6-13x^4z^8-32x^2y^2z^8\\&+111y^4z^8+99x^2z^{10}+27y^2z^{10}+44z^{12}=0.\end{aligned}$$

The ideal $\overline{I}$ of cusps has the minimal resolution

$$0\to\overline{S}(-9)^4\to\overline{S}(-8)\oplus\overline{S}(-7)^4\to\overline{S}\to\overline{S}/\overline{I}\to 0.$$

The Alexander polynomial is trivial.

(6.12) Covering ramified along b_1, b_2 and ℓ_1. This yields the curve $C_{6.12}$ defined over $\mathbb{Q}(\xi)$, where ξ a primitive third root of unity, by

$$\begin{aligned}
&-27x^{12} + (84\xi + 42)x^{10}y^2 + 135x^8y^4 + (-72\xi - 36)x^6y^6 - 45x^4y^8 + (-12\xi - 6)x^2y^{10}\\
&+ y^{12} + 216x^{10}z^2 + (-576\xi - 288)x^8y^2z^2 - 432x^6y^4z^2 + (192\xi + 96)x^4y^6z^2 + 24x^2y^8z^2\\
&- 684x^8z^4 + (1056\xi + 528)x^6y^2z^4 + 408x^4y^4z^4 + (-96\xi - 48)x^2y^6z^4 - 12y^8z^4 + 1088x^6z^6\\
&+ (-768\xi - 384)x^4y^2z^6 - 192x^2y^4z^6 - 912x^4z^8 + (192\xi + 96)x^2y^2z^8 + 48y^4z^8 + 384x^2z^{10}\\
&- 64z^{12} = 0,
\end{aligned}$$

whose Jacobian ideal J has the free resolution

$$0 \to S(-25) \oplus S(-23) \to S(-22) \oplus S(-20)^2 \oplus S(-19) \to S(-11)^3 \to S \to S/J \to 0.$$

This is an example of a degree 12 curve with 32 cusps and Alexander polynomial $t^2 - t + 1$.

(6.13) Covering ramified along b_1, b_2 and another inflectional tangent. This is the final example coming from the non-conical six-cuspidal sextic. For the last time, consider the reduction modulo 229. The curve $\overline{C_{6.13}}^{229}$ has 34 cusps and the equation

$$\begin{aligned}
&-104x^{12} - 105x^{10}y^2 - 82x^8y^4 + 11x^6y^6 - 44x^4y^8 - 30x^2y^{10} + 108y^{12} - 39x^{10}z^2\\
&+ 27x^8y^2z^2 + 61x^6y^4z^2 + 19x^4y^6z^2 - 26x^2y^8z^2 + 47y^{10}z^2 + 30x^8z^4 + 53x^6y^2z^4 - 42x^4y^4z^4\\
&- 81x^2y^6z^4 + 40y^8z^4 - 38x^6z^6 - 73x^4y^2z^6 + 4x^2y^4z^6 + 89y^6z^6 - 114x^4z^8 + 85x^2y^2z^8\\
&+ 29y^4z^8 - 96x^2z^{10} + 30y^2z^{10} + 11z^{12} = 0.
\end{aligned}$$

The ideal $\overline{I}$ of cusps in the ring $\overline{S} := \mathbb{F}_{229}[x, y, z]$ has the minimal resolution

$$0 \to \overline{S}(-10) \oplus \overline{S}(-9)^3 \to \overline{S}(-8)^2 \oplus \overline{S}(-7)^3 \to \overline{S} \to \overline{S}/\overline{I} \to 0,$$

which shows that the exponent s of $t^2 - t + 1$ in $\Delta_{C_{6.13}}(t)$ is at most one. Denote by Σ the set of cusps of $C_{6.13}$. Let Σ_1 be set of the 32 cusps coming from the base curve and the two inflectional bitangents and define $\Sigma_2 := \Sigma \setminus \Sigma_1$. Then

$$h_{S/I(\Sigma)}(7) = h_{S/I(\Sigma_1)}(7) + h_{S/I(\Sigma_2)}(7) - h_{S/(I(\Sigma_1)+I(\Sigma_2))}(7) \leq h_{S/I(\Sigma_1)}(7) + h_{S/I(\Sigma_2)}(7).$$

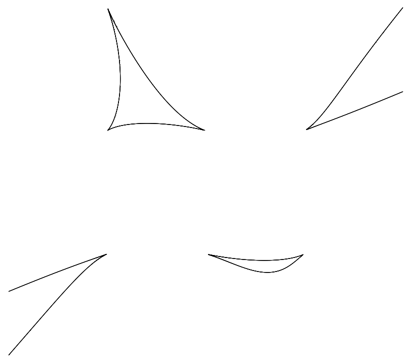
By the preceding example, $h_{S/I(\Sigma_1)}(7) = 31$. Σ_2 is a set of two points, hence a minimal graded free resolution of $S/I(\Sigma_2)$ looks like

$$0 \to S(-3) \to S(-2) \oplus S(-1) \to S \to S/I(\Sigma_2) \to 0$$

and $h_{S/I(\Sigma_2)}(7) = 2$. As a consequence,

$$s = p_{S/I(\Sigma)}(7) - h_{S/I(\Sigma)}(7) \geq 34 - 31 - 2 = 1.$$

That is, $\Delta_{C_{6.13}}(t) = t^2 - t + 1$.



A seven-cuspidal sextic whose cusps are all real:
$$-4x^2 + 12x^3 - 13x^4 + 6x^5 - x^6 + 24xy - 40x^2y + 18x^3y - 2x^4y - 36y^2$$
$$-48xy^2 + 93x^2y^2 - 36x^3y^2 + 2x^4y^2 + 108y^3 - 6xy^3 - 52x^2y^3 + 18x^3y^3$$
$$-117y^4 + 48xy^4 - x^2y^4 + 54y^5 - 18xy^5 - 9y^6 = 0$$

Sextic with seven cusps

Consider the curve C defined by the vanishing of

$$\begin{aligned}
&- 12x^5y - 40x^3y^3 - 12xy^5 + 6x^5z + 30x^4yz + 60x^3y^2z + 60x^2y^3z + 30xy^4z + 6y^5z - 3x^4z^2 \\
&- 108x^3yz^2 - 18x^2y^2z^2 - 108xy^3z^2 - 3y^4z^2 + 36x^3z^3 + 108x^2yz^3 + 108xy^2z^3 + 36y^3z^3 \\
&- 15x^2z^4 - 222xyz^4 - 15y^2z^4 + 54xz^5 + 54yz^5 - 17z^6 = 0.
\end{aligned}$$

This equation is given in [25, Formula (4.9)]. C is an irreducible sextic with seven cusps as the only singularities. Its Alexander polynomial equals $t^2 - t + 1$. Two inflectional tangents are given by $t_1 := V(x-y+2az)$ and $t_2 := V(x-y-2bz)$, where $a^4+3a^2+3 = b^4+3b^2+3 = 0$ and $b \notin \{a, -a\}$. Define further $\ell_1 := V(x + y + z)$ and $\ell_2 := V(x + 2y + z)$. These are lines intersecting C transversely.

(6.14) General covering. This gives a curve of degree 12 with 28 cusps and Alexander polynomial $t^2 - t + 1$.

(6.15) Covering ramified along t_1, ℓ_1 and ℓ_2. The resulting curve $C_{6.15}$ has 30 cusps and is given by

$$
\begin{aligned}
&-131x^{12} + (-132a + 2424)x^{10}y^2 + (3516a^2 - 1536a - 17796)x^8y^4 + (-13280a^3 - 22272a^2\\
&+ 20352a + 67792)x^6y^6 + (51456a^3 - 40464a^2 - 81600a - 244428)x^4y^8 + (72960a^3\\
&+ 175200a^2 + 283800a + 421836)x^2y^{10} + (-20576a^3 - 45804a^2 - 99420a - 52201)y^{12}\\
&+ (600a - 1872)x^{10}z^2 + (-6480a^2 - 3840a + 27120)x^8y^2z^2 + (29376a^3 + 61824a^2 - 14016a\\
&- 153600)x^6y^4z^2 + (-157440a^3 + 52992a^2 + 158400a + 707424)x^4y^6z^2 + (-198528a^3\\
&- 519552a^2 - 886272a - 1506480)x^2y^8z^2 + (52608a^3 + 163776a^2 + 380976a + 283800)y^{10}z^2\\
&+ (2160a^2 + 3840a - 10320)x^8z^4 + (-19584a^3 - 48384a^2 - 14208a + 115968)x^6y^2z^4\\
&+ (165888a^3 - 1152a^2 - 78336a - 761472)x^4y^4z^4 + (224256a^3 + 582912a^2 + 1063296a\\
&+ 2119872)x^2y^6z^4 + (-43008a^3 - 208512a^2 - 575232a - 573360)y^8z^4 + (4352a^3 + 10752a^2\\
&+ 9984a - 29184)x^6z^6 + (-73728a^3 - 13824a^2 - 12288a + 365568)x^4y^2z^6 + (-138240a^3\\
&- 319488a^2 - 602112a - 1485312)x^2y^4z^6 + (11264a^3 + 112128a^2 + 432384a + 583808)y^6z^6\\
&+ (12288a^3 + 2304a^2 + 12288a - 66048)x^4z^8 + (46080a^3 + 92160a^2 + 153600a\\
&+ 522240)x^2y^2z^8 + (-21504a^2 - 162816a - 327936)y^4z^8 + (-6144a^3 - 12288a^2 - 12288a\\
&- 73728)x^2z^{10} + (24576a + 98304)y^2z^{10} - 12288z^{12} = 0.
\end{aligned}
$$

Let J be the Jacobian ideal. Here, computing a minimal resolution of J^{sat} is faster than computing a resolution of J. The result is

$$0 \to S(-14) \oplus S(-13) \to S(-10) \oplus S(-9) \oplus S(-8) \to S \to S/J^{\mathrm{sat}} \to 0.$$

By Corollary 4.9, $\Delta_{C_{6.15}}(t) = t^2 - t + 1$.

(6.16) Covering ramified along t_1, t_2 and ℓ_1. The resulting curve $C_{6.16}$ has 32 cusps. Since its minimal field of definition is too large, work modulo 37. Then the reduction $\overline{C_{6.16}}^{37}$ has still 32 cusps and no further singularities and is given by the polynomial

$$
\begin{aligned}
&3x^{12} - 13x^{10}y^2 - 9x^8y^4 + 5x^4y^8 + 12x^2y^{10} + 3y^{12} + 15x^{10}z^2 - x^8y^2z^2 + 2x^6y^4z^2 - 2x^4y^6z^2\\
&+ x^2y^8z^2 - 15y^{10}z^2 + 12x^8z^4 - 11x^6y^2z^4 - 2x^4y^4z^4 - 11x^2y^6z^4 + 12y^8z^4 - 16x^6z^6\\
&+ 11x^4y^2z^6 - 11x^2y^4z^6 + 16y^6z^6 - 17x^4z^8 - 3x^2y^2z^8 - 17y^4z^8 + 10x^2z^{10} - 10y^2z^{10} - z^{12}.
\end{aligned}
$$

The ideal $\overline{I} \trianglelefteq \overline{S} := \mathbb{F}_{37}[x, y, z]$ of cusps of $\overline{C_{6.16}}^{37}$ has the minimal resolution

$$0 \to \overline{S}(-10) \oplus \overline{S}(-9)^2 \to \overline{S}(-8) \oplus \overline{S}(-7)^2 \oplus \overline{S}(-6) \to \overline{S} \to \overline{S}/\overline{I} \to 0.$$

As in (6.4), $\Delta_{C_{6.16}}(t) = t^2 - t + 1$.

(6.17) Covering ramified along three inflectional tangents. Take t_1 and t_2 as above. Another inflectional tangent t_3 is defined by $V(\alpha x+\beta y+\gamma z)$ where, the minimal polynomials of α, β, γ are always of degree 12. Thus the curve $C_{6.17}$ is not defined over a computationally feasible field extension of $\mathbb{Q}$. Again, work in positive characteristic. It turns out that modulo 349, the reduction of $C_{6.17}$ has 34 cusps and is given by

$$\begin{aligned}
&107x^{12} + 11x^{10}y^2 + 162x^8y^4 - 23x^6y^6 - 130x^4y^8 - 18x^2y^{10} + 174y^{12} - 64x^{10}z^2 + 6x^8y^2z^2 \\
&- 118x^6y^4z^2 + 152x^4y^6z^2 - 77x^2y^8z^2 - 108y^{10}z^2 - 145x^8z^4 - 69x^6y^2z^4 + 76x^4y^4z^4 \\
&- 133x^2y^6z^4 + 163y^8z^4 - 19x^6z^6 - 114x^4y^2z^6 + 167x^2y^4z^6 + 162y^6z^6 + 80x^4z^8 - 70x^2y^2z^8 \\
&+ 168y^4z^8 + 150x^2z^{10} - 22y^2z^{10} + 139z^{12} = 0.
\end{aligned}$$

A minimal free resolution of the ideal $\overline{I} \trianglelefteq \overline{S} := \mathbb{F}_{349}[x, y, z]$ of cusps is given by

$$0 \to \overline{S}(-10) \oplus \overline{S}(-9)^3 \to \overline{S}(-8)^2 \oplus \overline{S}(-7)^3 \to \overline{S} \to \overline{S}/\overline{I} \to 0.$$

Hence with the same reasoning as in (6.4), $\Delta_{C_{6.17}}(t) = t^2 - t + 1$.

Sextic with eight cusps

Consider the curve C defined by the polynomial (compare [25, Formula (4.10)])

$$\begin{aligned}
&745x^6 - 4278x^5y + 11175x^4y^2 - 14260x^3y^3 + 11175x^2y^4 - 4278xy^5 + 745y^6 + 120x^5z \\
&- 264x^4yz - 1392x^3y^2z - 1392x^2y^3z - 264xy^4z + 120y^5z - 1866x^4z^2 + 6792x^3yz^2 \\
&- 6012x^2y^2z^2 + 6792xy^3z^2 - 1866y^4z^2 + 144x^3z^3 - 2160x^2yz^3 - 2160xy^2z^3 + 144y^3z^3 \\
&+ 1521x^2z^4 + 126xyz^4 + 1521y^2z^4 - 648xz^5 - 648yz^5.
\end{aligned}$$

This is an eight-cuspidal sextic with $\Delta_C(t) = (t^2 - t + 1)^2$. The following inflectional tangents will be used:

$$\begin{aligned}
t_{1/2} &:= V(x - y \pm 2\sqrt{2}/3z), \\
t_3 &:= V\left(x + y \cdot \frac{1}{67}\left(-77 - 18i\sqrt{3}\right) + z \cdot \frac{1}{67}\left(-51 + 2i\sqrt{3}\right)\right).
\end{aligned}$$

Further choose the following lines intersecting C transversely:

$$\ell_1 := V(x + y + z), \quad \ell_2 := V(x + y + 2z).$$

(6.18) General covering. This gives a degree 12 curve with 32 cusps and Alexander polynomial $(t^2 - t + 1)^2$.

(6.19) One inflectional tangent. Pick t_1 as inflectional tangent and the two lines ℓ_1, ℓ_2 for the covering. This produces the curve $C_{6.19}$ defined over $\mathbb{Q}(\sqrt{2})$ with the equation

$$\begin{aligned}
&729x^{12} + 2916\sqrt{2}x^{10}y^2 + 8262x^8y^4 + 4752\sqrt{2}x^6y^6 - 4023x^4y^8 - 7668\sqrt{2}x^2y^{10} + 11448y^{12} \\
&- 2916\sqrt{2}x^{10}z^2 - 16524x^8y^2z^2 - 14256\sqrt{2}x^6y^4z^2 + 10260x^4y^6z^2 + 30564\sqrt{2}x^2y^8z^2 \\
&- 45576y^{10}z^2 + 8262x^8z^4 + 14256\sqrt{2}x^6y^2z^4 - 8586x^4y^4z^4 - 48168\sqrt{2}x^2y^6z^4 + 75528y^8z^4 \\
&- 4752\sqrt{2}x^6z^6 + 2268x^4y^2z^6 + 37512\sqrt{2}x^2y^4z^6 - 66680y^6z^6 + 81x^4z^8 - 14436\sqrt{2}x^2y^2z^8 \\
&+ 33072y^4z^8 + 2196\sqrt{2}x^2z^{10} - 8736y^2z^{10} + 960z^{12} = 0.
\end{aligned}$$

The Jacobian ideal J has the minimal resolution

$$\begin{aligned}0 &\to S(-24)\oplus S(-23)^2\oplus S(-21)\to S(-22)^3\oplus S(-20)\oplus S(-19)^2\\ &\to S(-11)^3\to S\to S/J\to 0.\end{aligned}$$

Hence $C_{6.19}$ is an example of a degree 12 curve with 34 cusps and Alexander polynomial $(t^2-t+1)^2$.

(6.20) Two inflectional tangents. The covering ramified along t_1, t_2 and ℓ_1 yields the curve $C_{6.20}$ defined via

$$\begin{aligned}&21870x^{12}-264627x^{10}y^2+1001646x^8y^4-771282x^6y^6+1001646x^4y^8-264627x^2y^{10}\\ &+21870y^{12}-90396\sqrt{2}x^{10}z^2+731916\sqrt{2}x^8y^2z^2-1743768\sqrt{2}x^6y^4z^2+1743768\sqrt{2}x^4y^6z^2\\ &-731916\sqrt{2}x^2y^8z^2+90396\sqrt{2}y^{10}z^2+307152x^8z^4-1601856x^6y^2z^4+2589408x^4y^4z^4\\ &-1601856x^2y^6z^4+307152y^8z^4-275616\sqrt{2}x^6z^6+909792\sqrt{2}x^4y^2z^6-909792\sqrt{2}x^2y^4z^6\\ &+275616\sqrt{2}y^6z^6+276480x^4z^8-552960x^2y^2z^8+276480y^4z^8-73728\sqrt{2}x^2z^{10}\\ &+73728\sqrt{2}y^2z^{10}+16384z^{12}=0.\end{aligned}$$

with 36 cusps and Alexander polynomial $\Delta_{C_{6.20}}(t)=(t^2-t+1)^3$. The minimal resolution for the Jacobian ideal J is given by

$$0\to S(-23)^3\oplus S(-21)\to S(-22)^3\oplus S(-19)^3\to S(-11)^3\to S\to S/J\to 0.$$

(6.21) Three inflectional tangents. Pick t_1, t_2 and t_3 for the covering. The resulting curve $C_{6.21}$ has a very long equation, which will not be displayed here. However, $C_{6.21}$ is defined over $\mathbb{Q}(\sqrt{2}, i\sqrt{3})$, and it is still fesible to compute a minimal resolution of the saturation J^{sat} of the Jacobian ideal:

$$0\to S(-14)^3\oplus S(-13)\oplus S(-12)^2\to S(-12)^2\oplus S(-11)^5\to S\to S/J^{\text{sat}}\to 0$$

Thus $C_{6.21}$ is an example for a cuspidal curve of degree 12 with 38 cusps and Alexander polynomial $(t^2-t+1)^3$.

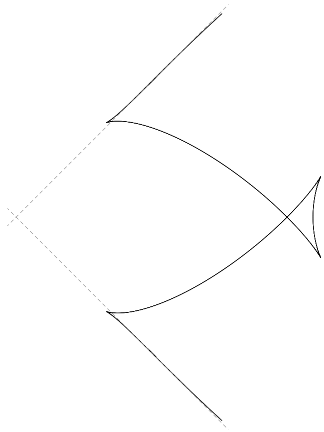
Nodal eight-cuspidal sextic

In the definition of cuspidal curve, also ordinary double points were allowed as singularities. However, the presence of nodes does not seem to influence the Alexander polynomial. As an example, some Kummer coverings of a nodal eight-cuspidal sextic are presented here.

Consider the two-cuspidal quartic presented in the previous section. Cancelling the factor 2 before y^2 yields also a two-cuspidal quartic C'. The dual curve C is an irreducible eight-cuspidal sextic with additionally one ordinary double point at $(1:0:1)$. C is given by

$$\begin{aligned}&8x^4y^2-3x^2y^4+11y^6-32x^5z-54xy^4z+64x^4z^2\\ &+24x^2y^2z^2+21y^4z^2-48x^2z^4-48y^2z^4+16z^6=0.\end{aligned}$$

Due to the presence of the node, C has only two inflection points by the Plücker formulas. These are situated at $(0 : \pm 1 : 1)$, the corresponding inflectional tangents are $t_{1/2} := V(x \pm y + z)$. Pick the two transversely intersecting lines $\ell_1 := 2y - z$ and $\ell_2 := x - 2y + z$.



The nodal eight-cuspidal sextic C and t_1, t_2 in the affine chart $z = 1$

(6.22) General covering. This yields a curve of degree 12 with 4 nodes and 32 cusps with Alexander polynomial $(t^2 - t + 1)^2$.

(6.23) Covering ramified along t_1, ℓ_1 and ℓ_2. This produces the curve

$$\begin{aligned}&13x^{12} - 90x^{10}y^2 + 30x^8y^4 + 520x^6y^6 - 243x^4y^8 - 1398x^2y^{10} + 1168y^{12} - 192x^{10}z^2\\&+ 528x^8y^2z^2 + 1248x^6y^4z^2 - 1824x^4y^6z^2 - 6432x^2y^8z^2 + 7536y^{10}z^2 + 696x^8z^4 + 96x^6y^2z^4\\&- 3096x^4y^4z^4 - 11856x^2y^6z^4 + 20208y^8z^4 - 736x^6z^6 - 1392x^4y^2z^6 - 11136x^2y^4z^6\\&+ 28816y^6z^6 + 144x^4z^8 - 5472x^2y^2z^8 + 23040y^4z^8 - 1152x^2z^{10} + 9792y^2z^{10} + 1728z^{12} = 0\end{aligned}$$

with 4 nodes and 34 cusps. The minimal graded free resolution of the ideal I of the 34 cusps is given by

$$0 \to S(-10)^2 \oplus S(-9) \to S(-8)^2 \oplus S(-7) \oplus S(-6) \to S \to S/I \to 0,$$

so the Alexander polynomial equals $(t^2 - t + 1)^2$.

(6.24) Covering ramified along t_1, t_2 and ℓ_1. The result is the curve given by

$$\begin{aligned}&28x^{12} - 235x^{10}y^2 + 2304x^8y^4 - 10818x^6y^6 + 22896x^4y^8 - 22059x^2y^{10} + 7884y^{12} - 328x^{10}z^2\\&+ 3096x^8y^2z^2 - 20112x^6y^4z^2 + 59760x^4y^6z^2 - 75816x^2y^8z^2 + 33912y^{10}z^2 + 1552x^8z^4\\&- 14048x^6y^2z^4 + 60000x^4y^4z^4 - 104544x^2y^6z^4 + 60624y^8z^4 - 3808x^6z^6 + 27872x^4y^2z^6\\&- 72480x^2y^4z^6 + 57632y^6z^6 + 5120x^4z^8 - 25344x^2y^2z^8 + 30720y^4z^8 - 3584x^2z^{10}\\&+ 8704y^2z^{10} + 1024z^{12} = 0.\end{aligned}$$

This is an irreducible degree 12 curve with 4 nodes and 36 cusps. Its Alexander polynomial is $(t^2-t+1)^3$, as can be seen from the minimal graded free resolution of the ideal I of cusps:

$$0 \to S(-10)^3 \to S(-8)^3 \oplus S(-6) \to S \to S/I \to 0.$$

Thus these curves have the same Alexander polynomial as the coverings of the eight-cuspidal sextic without nodes. The same phenomenon happens when considering coverings from other cuspidal curves with nodes, e. g. nodal two-cuspidal quartics or seven-cuspidal sextics with two nodes. The corresponding computations are omitted here.

Sextic with nine cusps

Finally consider the curve C given by

$$x^6 + y^6 + z^6 - 2(x^3y^3 + x^3z^3 + y^3z^3) = 0.$$

This is a nine-cuspidal sextic. Any nine-cuspidal sextic is dual to a smooth cubic, and hence has no inflection points at all. Thus only the general covering may be applied to obtain a cuspidal curve of degree 12.

(6.25) General covering. The minimal graded free resolution of the ideal I of the nine cusps of C is

$$0 \to S(-5)^3 \to S(-4)^3 \oplus S(-3) \to S \to S/I \to 0.$$

Hence $\Delta_C(t) = (t^2 - t + 1)^3$. This is also the Alexander polynomial of the 36-cuspidal curve obtained via a general covering of degree two.

5.4 Concluding remarks

Limitations of the strategy. The Kummer coverings of quartics and sextics produce many good examples of cuspidal curves of degree 12. However, these curves are very special. For example, it is impossible to obtain curves whose number of cusps is neither divisible by two or three.

Furthermore, the choice of the base curves and the lines involved in the cover may influence the Alexander polynomial. Based on numerous computations, this does not seem to be the case. For example, all three-cuspidal quartics are projectively equivalent, as this holds for their dual curves, which are nodal cubics. Moreover, if a transversely intersecting line is replaced by another such line, then the preimages of the two base curves under the Kummer covering are equivalent with respect to equisingular deformation. That is, there exists a path in some equisingular stratum in S_{12} connecting the two curves. This implies that the complements are homeomorphic and thus the Alexander polynomial does not change ([6]).

An application (compare [3, Section 4]). Define $\mathcal{C}_{34,2} \subseteq S_{12}$ to be the equisingular deformation space (see [17] for a precise definition) of the 34-cuspidal curve $C_{6.19}$ with Alexander polynomial $(t^2 - t + 1)^2$. This space has codimension 68 in S_{12}:

Let $f \in \mathbb{C}[x, y, z]_{12}$ be an equation for the curve $C_{6.19}$. [17, Proposition 2.5] states that the Zariski tangent space of $\mathcal{C}_{34,2}$ at $C_{6.19}$ has codimension $h_{S/J(f)^{\text{sat}}}(12)$ in S_{12}. By Corollary 4.9, the codimension is thus $h_{S/J(f)^{\text{sat}}}(12) = p_{S/J(f)^{\text{sat}}}(12) = 68$. Hence the codimension of the space $\mathcal{C}_{34,2}$ itself is at least 68. Any curve in this space has 34 cusps, which are double points. This shows that also $\operatorname{codim}_{S_{12}} \mathcal{C}_{34,2} \leq 2 \cdot 34 = 68$.

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Summary

Let $C \subseteq \mathbb{P}^2_{\mathbb{C}}$ be a reduced curve in the complex projective plane. Then an invariant of the fundamental group $\pi_1(\mathbb{P}^2_{\mathbb{C}} \setminus C)$ is given by the Alexander polynomial $\Delta_C(t) \in \mathbb{Q}[t]$ of C. If C is irreducible of degree $6k$ for some $k \in \mathbb{N}$ and has only A_1 and A_2 singularities, then

$$\Delta_C(t) = (t^2 - t + 1)^s, \quad \text{where } s \in \mathbb{N}_0 \quad \text{and} \quad s \leq \frac{1}{4}\left(15k - 1 - \sqrt{15k^2 - 18k + 7}\right).$$

The Alexander polynomial has many other interpretations:

Theorem. *Let $f \in \mathbb{C}[x,y,z]_{6k}$ be a homogeneous irreducible polynomial of degree $6k$ for some $k \in \mathbb{N}$ such that $C := V(f)$ is a singular curve and each singular point is either of type A_1 or of type A_2. Denote by Σ the set of ordinary cusps of C. Then the following numbers coincide:*

(a) The degree of the Alexander polynomial $\Delta_C(t)$,

(b) the irregularity of a desingularization of the cyclic multiple plane $u^{6k} = f(x,y,z)$,

(c) the rank of the Mordell-Weil group of an elliptic threefold birational to the hypersurface defined by $-v^2 + u^3 + f(x,y,z) = 0$ in the weighted projective space $\mathbb{P}^4_{\mathbb{C}}(2k, 3k, 1, 1, 1)$,

(d) the rank of the group of quasi-toric relations of type $(2,3,6)$ of f,

(e) $2\dim_{\mathbb{C}} \operatorname{coker}\varphi$, where

$$\varphi : \mathbb{C}[x,y,z]_{5k-3} \to \mathbb{C}^{\#\Sigma}, \quad f \mapsto (f(p))_{p\in\Sigma},$$

(f) $2\dim_{\mathbb{C}} \operatorname{coker}\psi$, where

$$\psi : \mathbb{C}[x,y,z]_{7k-3} \to \mathbb{C}^{2\#\Sigma}, \quad f \mapsto \left(f(p), \frac{\partial f}{\partial \ell_p}(p)\right)_{p\in\Sigma}$$

and $\ell_p = 0$ is a square-free equation for the cuspidal tangent at $p \in \Sigma$,

(g) the difference between $\#\Sigma$ and the Hilbert polynomial of $S/I(\Sigma)$ evaluated at $5k-3$,

(h) the difference between $2\#\Sigma$ and the Hilbert polynomial of $S/J(f)^{\mathrm{sat}}$ evaluated at $7k-3$, where $J(f)$ denotes the Jacobian ideal of f,

(i) $\#\{i \in \{0,\dots,t\} \mid b_i = 5k\}$, where

$$0 \to \bigoplus_{i=1}^{t} S(-b_i) \to \bigoplus_{i=1}^{t+1} S(-a_i) \to S \to S/I(\Sigma) \to 0$$

is a minimal graded free resolution of $S/I(\Sigma)$,

(j) $\#\{i \in \{0, \ldots, t\} \mid b_i = 7k\}$, *where*

$$0 \to \bigoplus_{i=1}^{t} S(-b_i) \to \bigoplus_{i=1}^{t+1} S(-a_i) \to S \to S/J(f)^{\mathrm{sat}} \to 0$$

is a minimal graded free resolution of $S/J(f)^{\mathrm{sat}}$,

(k) $\#\{i \in \{0, \ldots, t+2\} \mid b_i = 11k - 3\}$, *where*

$$0 \to \bigoplus_{i=1}^{t} S(-c_i) \to \bigoplus_{i=1}^{t+2} S(-b_i) \to S(-6k+1)^3 \to S \to S/J(f) \to 0$$

is a minimal graded free resolution of $S/J(f)$.

Except for one case, it is possible to determine the Alexander polynomial from the number of cusps of some degree $6k$ curve C if $k = 1$. For $k = 2$, this turns out more difficult. In this diploma thesis, in total 37 cuspidal curves of degree 12 and their Alexander polynomials are computed to gain more insight into this matter.