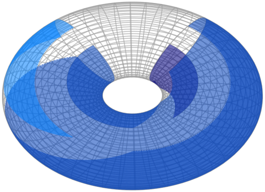
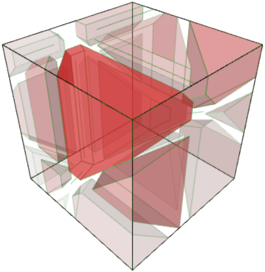


On the tropical and zonotopal geometry of periodic timetabling



Enrico Bortoletto, Niels Lindner, Berenike Masing
Zuse Institute Berlin

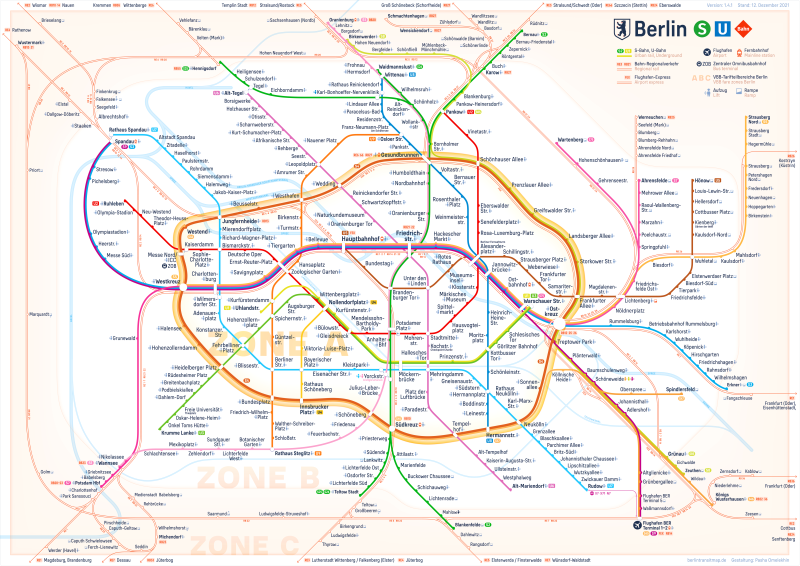
Research Seminar on Discrete and Convex Geometry
@ TU Berlin

May 18, 2022

Part 1

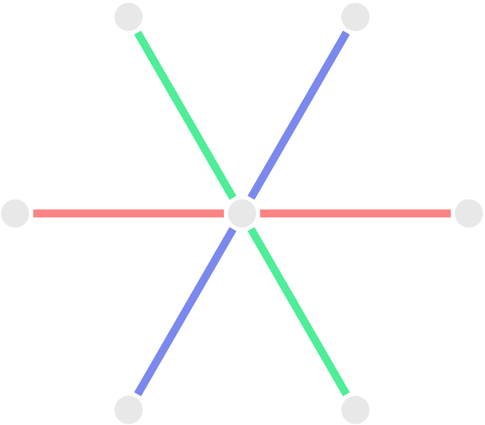
Periodic Timetabling in Public Transport

A Line Network



berlintransitmap.de

From Line Networks to Event-Activity Networks



Line Network, 3 bidirectional lines

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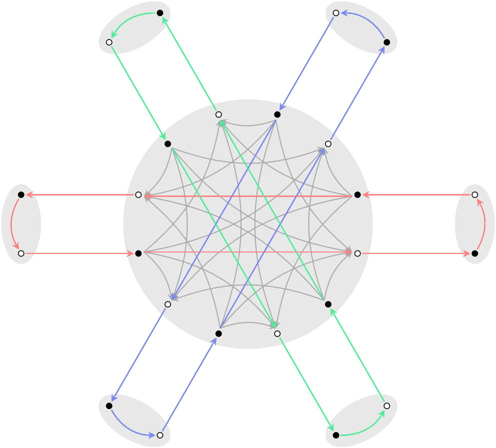

Event-Activity Network

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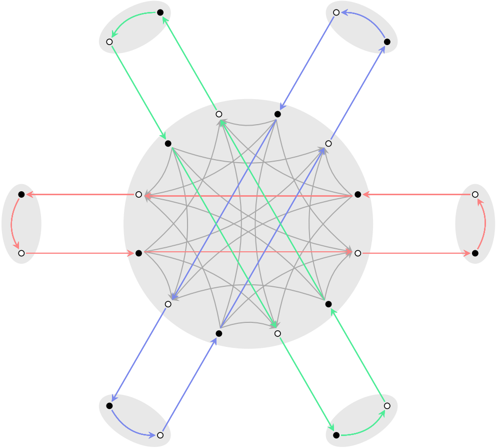


Event-Activity Network

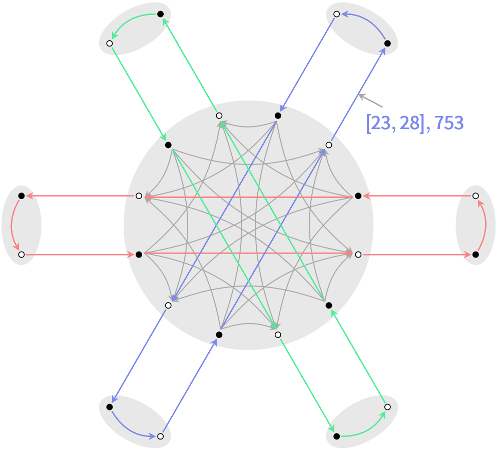
Events:

- ● arrival
- ○ departure

Activities:

- → drive, dwell, turn
- → transfer
- . . .

Periodic Timetabling in Public Transport



Periodic Event Scheduling Instance

Bounds:

- driving times
- minimum transfer times
- maximum dwell times
- minimum headway times
- . . .

Weights:

- passenger load
- turnaround penalties
- . . .

Period time:

- e.g., $T = 60$ for 1 hour, resolution of 1 minute

Periodic Event Scheduling Problem (PESP)

Given

$G = (V, A)$	event-activity network,
$T \in \mathbb{N}$	period time,
$\ell \in \mathbb{R}^A$	lower bounds,
$u \in \mathbb{R}^A$	upper bounds,
$w \in \mathbb{R}^A_{\geq 0}$	weights,

find

$\pi \in [0, T)^V$	periodic timetable,
$x \in \mathbb{R}^A$	periodic tension

such that

(1) $\pi_j - \pi_i \equiv x_{ij} \mod T$ for all $ij \in A$,

(2) $\ell \leq x \leq u$,

(3) $w^\top x$ is minimum,

or decide that no such (π, x) exists.

(Serafini and Ukovich, 1989)

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Incidence-based MIP formulation:

$$\begin{array}{lrl}
\text{Minimize} & w^\top x & \\
\text{s.t.} & \pi_j - \pi_i = x_{ij} - Tp_{ij}, & ij \in A, \\
& \ell_{ij} \leq x_{ij} \leq u_{ij}, & ij \in A, \\
& \pi_i \in \mathbb{R}, & i \in V, \\
& p_{ij} \in \mathbb{Z}, & ij \in A.
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$p \in \mathbb{Z}^A$ periodic offsets

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Assumptions after preprocessing:

- G is weakly (2-)connected
- G has no arc $a \in A$ with $\ell_a = u_a$
- $0 \leq \ell < T$ and $0 \leq u - \ell < T$

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$p \in \mathbb{Z}^A$ periodic offsets

Redundancy among periodic offsets p:

- could impose $0 \leq \pi_i < T$ and $p_{ij} \in \{0, 1, 2\}$
- could set $p_{ij} = 0$ along spanning forest

(Nachtigall, 1994, Liebchen, 2006)

Hardness of PESP

Theory:

- NP-hard for fixed $T \geq 3$
  (Odijk, 1994, Nachtigall, 1996)
- NP-hard if G has treewidth ≥ 2
  (L. and Reisch, 2020)
- NP-hard cutting plane separation (cycle, change-cycle, flip)
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Practice:

- rich literature on algorithms:
  - MIP
  - CP
  - SAT (also MaxSAT and SAT+ML)
  - Modulo Network Simplex
  - Matching, Merging, Maximum Cuts, Graph Partitioning, ...
- several success stories (Berlin, Copenhagen, Netherlands, Switzerland, ...)
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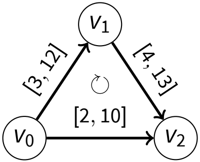
Summary: There are many ways to heuristically optimize periodic timetables, but it is hard to assess the actual quality.

Question: Can we get more insight by studying the geometry of periodic timetables?

Cycles



$$\gamma = (\gamma_{v_0 v_1}, \gamma_{v_0 v_2}, \gamma_{v_1 v_2}) = (1, -1, 1)$$

oriented cycle:

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$\{a \in A \mid \gamma_a \neq 0\}$ is a cycle when ignoring orientations

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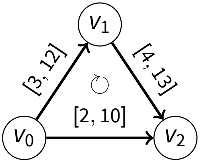

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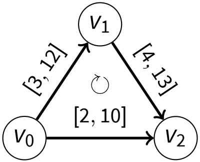

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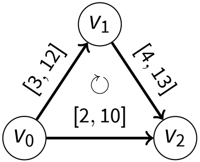

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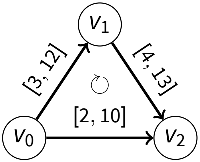

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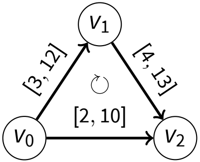

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cycle matrix of an integral cycle basis $\mathcal{B}$:
matrix $\Gamma \in \{-1, 0, 1\}^{\mathcal{B} \times A}$ with the vectors in $\mathcal{B}$ as rows

Cycle Periodicity

Theorem (Cycle periodicity property)

Let $G = (V, A)$ be a digraph with incidence matrix $B \in \{-1, 0, 1\}^{V \times A}$. Let Γ be the cycle matrix of an integral cycle basis $\mathcal{B}$ of G. Then, as $\mathbb{Z}$-modules,

$$\operatorname{im} B^\top = \ker \Gamma.$$

For friends of graph cohomology: The following sequence is exact:

$$0 \to \mathbb{Z}^C \to \mathbb{Z}^V \xrightarrow{B^T} \mathbb{Z}^A \xrightarrow{\Gamma} \mathbb{Z}^{\mathcal{B}} \to 0$$

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Corollary (Liebchen, Peeters, 2009)

Let $x \in \mathbb{R}^A$ and $T \in \mathbb{N}$. Then the following are equivalent:

(1) *There is a vector $\pi \in \mathbb{R}^V$ such that $x_{ij} \equiv \pi_j - \pi_i \mod T$ for all $ij \in A$.*

(2) $\Gamma x \equiv 0 \mod T$.

MIP Formulations and Spaces of Interest

MIP Formulations

Incidence-based MIP formulation:

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\begin{aligned}
\text{Minimize} \quad & w^\top x \\
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Cycle-based MIP formulation:

$$\begin{aligned} \text{Minimize} \quad & w^\top x \\ \text{s.t.} \quad & \Gamma x = Tz \\ & \ell \leq x \leq u \\ & z \in \mathbb{Z}^{\mathcal{B}} \end{aligned}$$

$z \in \mathbb{Z}^{\mathcal{B}}$ cycle offsets

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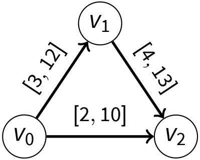
Timetabling Spaces

$X := \operatorname{conv}\left\{x \in \mathbb{R}^A \mid \exists z \in \mathbb{Z}^{\mathcal{B}} : \Gamma x = Tz, \ell \leq x \leq u\right\}$ convex hull of feas. periodic tensions

$\Pi := \{\pi \in \mathbb{R}^V \mid \exists p \in \mathbb{Z}^A : \ell \leq -B^\top \pi + Tp \leq u\}$ space of feasible periodic timetables

$Z := \{z \in \mathbb{R}^{\mathcal{B}} \mid \exists x \in \mathbb{R}^A : \Gamma x = Tz, \ell \leq x \leq u\}$ space of feasible cycle offsets

Gallery of Timetabling Spaces



PESP instance with $n = 3, m = 3, \mu = 1$

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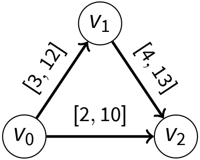

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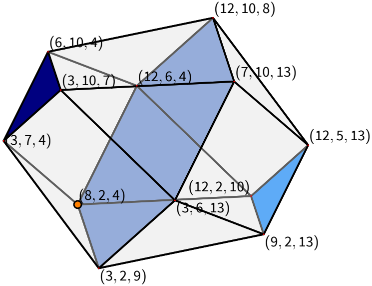

X is a polytope ⇝ standard toolbox of mixed-integer linear programming

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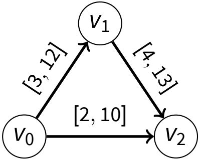

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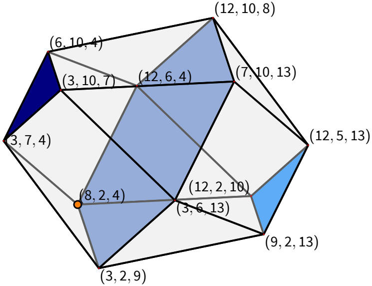

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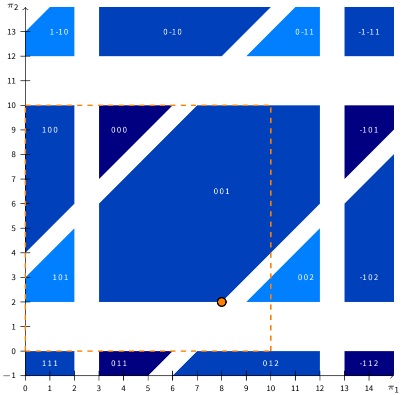

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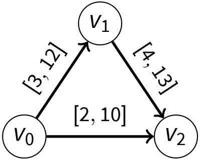

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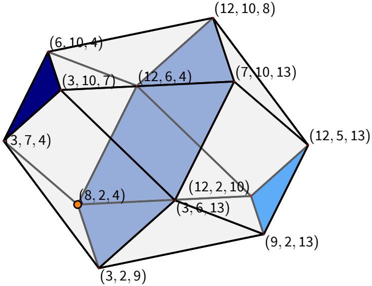

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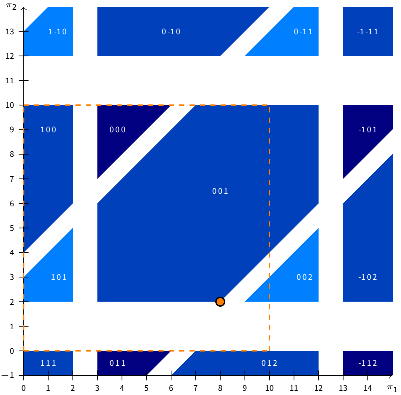

$\Pi/\mathbb{R}\mathbf{1}$ is periodically tiled by polyt(r)opes

Z is a zonotope

Part 2

The Tropical Tiling of the Space of Periodic Timetables

Decomposing the Space of Periodic Timetables

Decomposition

Recall that the space of feasible periodic timetables is

$$\Pi := \{\pi \in \mathbb{R}^V \mid \exists p \in \mathbb{Z}^A : \ell \leq -B^\top \pi + Tp \leq u\}.$$

The space Π decomposes into polyhedral regions:

$$\Pi = \bigcup_{p \in \mathbb{Z}^A} R(p), \quad \text{where } R(p) := \{\pi \in \mathbb{R}^V \mid \ell - Tp \leq -B^\top \pi \leq u - Tp\}.$$

Due to the assumption $0 \leq u - \ell < T$, the union is disjoint.

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Weighted Digraph Polyhedra

Add a reverse copy $\overline{a}$ of each arc a. This produces a new graph $\overline{G} = (\overline{V}, \overline{A})$ with $\overline{V} = V$. If we set $\kappa(p)_a := u_a - Tp_a$ and $\kappa(p)_{\overline{a}} := -\ell_a + Tp_a$, then

$$R(p) = \{\pi \in \mathbb{R}^{\overline{V}} \mid \pi_j - \pi_i \leq \kappa(p)_{ij} \text{ for all } ij \in \overline{A}\}.$$

This means that $R(p)$ is the *weighted digraph polyhedron* (Joswig, Loho, 2016) associated to $(\overline{G}, \kappa(p))$. In combinatorial optimization terms, $R(p)$ is the polyhedron of feasible potentials in $\overline{G}$ w.r.t. the arc costs $\kappa(p)$.

Decomposing the Space of Periodic Timetables

A First Symmetry

Since G was assumed to be weakly connected, $\overline{G}$ is strongly connected. This means by (Joswig, Loho, 2016):

- The recession cone of $R(p)$ is $\mathbb{R}\mathbf{1}$ (i.e., the kernel of $B^\top$).
- The quotient $R(p)/\mathbb{R}\mathbf{1}$ is a polytope.

Choosing coordinates on $R(p)/\mathbb{R}\mathbf{1}$ amounts to the periodic timetabler's wisdom that a timetable π can be fixed at one event $v_0 \in V$ to $\pi_{v_0} := 0$ without affecting feasiblity or optimality.

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Polytropes

A *polytrope* is the convex hull of finitely many points, both in the ordinary and the tropical sense. Polytropes are exactly the quotients of weighted digraph polyhedra of strongly connected digraphs by $\mathbb{R}\mathbf{1}$ (Joswig, Kulas, 2010).

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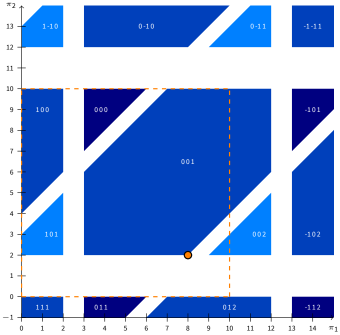
Corollary

The space $\Pi/\mathbb{R}\mathbf{1}$ decomposes into the disjoint union of the polytropes $R(p)/\mathbb{R}\mathbf{1}$.

The Periodic Timetabling Torus

Periodicity: If $\pi \in \Pi$, then $\pi + Tq \in \Pi$ for all $q \in \mathbb{Z}^V$. Consequently, we could consider the space of timetables inside the $(n-1)$-dimensional torus

$$\mathcal{T} := (\mathbb{R}^V/(T\mathbb{Z})^V)/\mathbb{R}\mathbf{1}.$$



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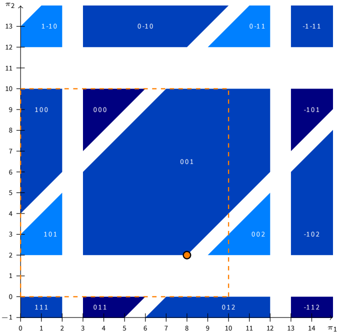

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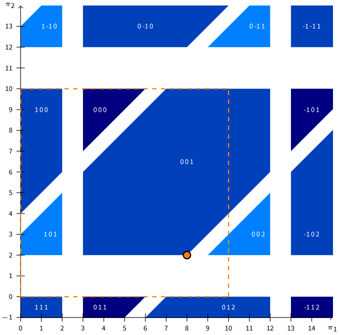
In our example,

$$z = \frac{\Gamma x}{T} \leq \left\lfloor \frac{12 - 2 + 13}{10} \right\rfloor = 2,$$

$$z = \frac{\Gamma x}{T} \geq \left\lceil \frac{3 - 10 + 4}{10} \right\rceil = 0,$$

so there are at most 3 non-empty polytropes on the torus (for $z \in \{0, 1, 2\}$).



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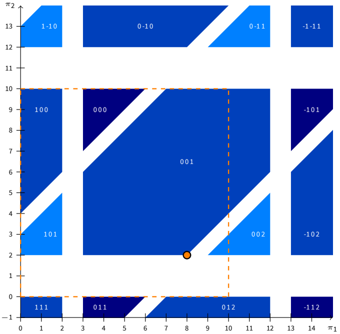

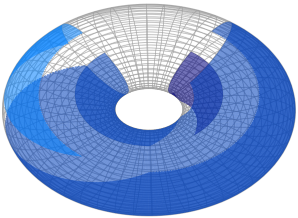

More on Timetabling Polytropes

Dimension

- $R(p) = \emptyset$ if and only if $\overline{G}$ contains a negative weight (directed) cycle w.r.t. $\kappa(p)$.
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Vertices

- Every vertex of $R(p)/\mathbb{R}\mathbf{1}$ corresponds to a unique spanning subgraph of $\overline{G}$.
- For each $i \in V$, the i-th tropical vertex of $R(p)/\mathbb{R}\mathbf{1}$ corresponds to a shortest path tree of $(\overline{G}, \kappa(p))$ rooted at i. (Joswig, Kulas, 2010).

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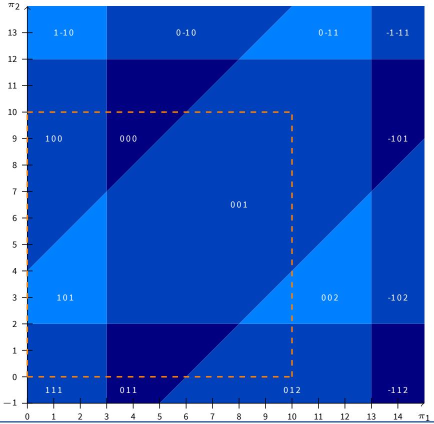
Relation to the Periodic Tension Polytope

- The map $m_p : \pi \mapsto -B^\top \pi + Tp$ embeds $R(p)/\mathbb{R}\mathbf{1}$ into X.
- X is the convex hull of $\{\mathrm{im}(m_p) \mid p \in \mathbb{Z}^A\}$.
- $\mathrm{im}(m_p)$ is the intersection of the affine space $\mathrm{im}(B^\top) + Tp$ with the LP relaxation polytope $X_{\mathrm{LP}} = \prod_{a \in A} [\ell_a, u_a]$ of X.

Polytropes in the Limit Instance

Let $R(p)/\mathbb{R}\mathbf{1}$ be a polytrope. The offset p also defines a polytrope $R'(p)/\mathbb{R}\mathbf{1}$ a of the "limit" instance where $u := \ell + T$. The union of the polytropes is then no longer disjoint and covers all of $\mathbb{R}^V/\mathbb{R}\mathbf{1}$.



Tropical Neighborhood Search

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Lemma

Let $p, p' \in \mathbb{Z}^A$. Then $R'(p)/\mathbb{R}\mathbf{1} \cap R'(p')/\mathbb{R}\mathbf{1} \neq \emptyset$ if and only if there is an arc $a \in A$ with $p = p' \pm e_a$. In this case, the polytropes intersect in a common face.

In particular, the $R'(p)$ give rise to a polyt(r)opal subdivision of $\mathbb{R}^V/\mathbb{R}\mathbf{1}$.

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Neighbors

We call $R(p)/\mathbb{R}\mathbf{1}$ and $R(p')/\mathbb{R}\mathbf{1}$ *neighbors* if $R'(p)/\mathbb{R}\mathbf{1}$ and $R'(p')/\mathbb{R}\mathbf{1}$ intersect in a common facet.

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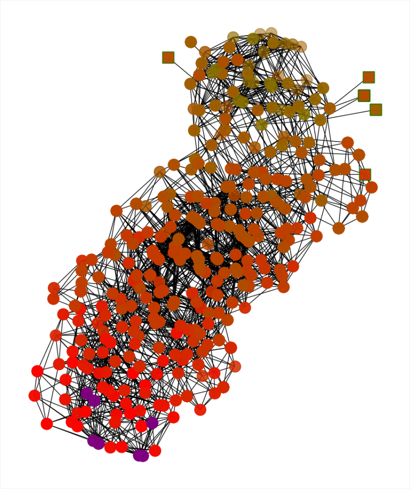
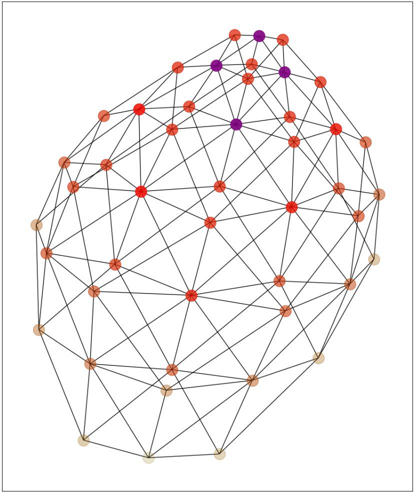
Tropical Neighborhood Search for Periodic Timetabling

Given a non-empty polytrope $R(p)/\mathbb{R}\mathbf{1}$, solve PESP on $R(p)/\mathbb{R}\mathbf{1}$ (this is a linear program, and dual to uncapacitated min cost flow). While there is an improving neighbor of $R(p)/\mathbb{R}\mathbf{1}$: Go to the best neighboring polytrope, and repeat.

Tropical Neighborhood Search



modulo network simplex search space
colored by objective value
squares are local non-global optima



tropical neighborhood search space
colored by objective value

Part 3

The Zonotope of Cycle Offsets

Zonotopes and Their Tilings

Zonotopes

A *zonotope* $Z(M, b)$ is the image of a (hyper)cube w.r.t. an affine map $x \mapsto Mx + b$. In particular, the space of feasible cycle offsets of a PESP instance with a chosen cycle basis $\mathcal{B}$ and a cycle matrix Γ,

$$Z := \{z \in \mathbb{R}^{\mathcal{B}} \mid \exists x \in \mathbb{R}^{A} : \Gamma x = Tz, \ell \leq x \leq u\} = \frac{1}{T}\Gamma(X_{\mathrm{LP}}) = \frac{1}{T}\Gamma\left(\prod_{a \in A}[\ell_a, u_a]\right),$$

is a μ-dimensional zonotope: We can take $M := \Gamma'$, where Γ' is obtained from Γ by scaling each column a by $\frac{u_a - \ell_a}{T}$, and $b := \frac{\Gamma \ell}{T}$. We call Z a *cycle offset zonotope*.

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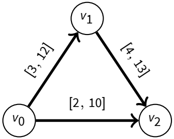
Zonotopal Tilings

A *zonotopal tiling* of a zonotope $Z(M, b)$ is a polyhedral subdivision of $Z(M, b)$ such that each cell is a zonotope $Z(M_S, b_S)$, where M_S is the submatrix of M with the colums indexed by S removed. We will call maximal cells *tiles*. A zonotopal tiling is *fine* if all tiles are parellelotopes, i.e., the columns of M_S are linearly independent.

Pictures of Cycle Offset Zonotopes



$Z \subseteq \mathbb{R}$ with 3 tiles and 3 lattice points

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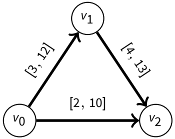

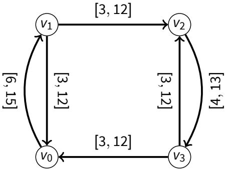

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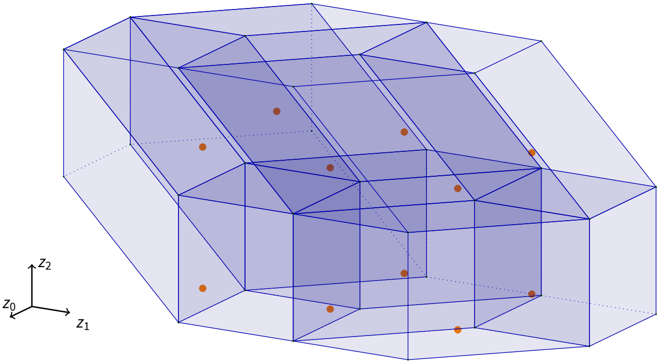

$Z \subseteq \mathbb{R}^3$ with 12 tiles and 11 lattice points

Zonotopal Tilings of Cycle Offset Zonotopes

Let Z be a cycle offset zonotope of a PESP instance.

Lemma

The k-dimensional cells of any fine zonotopal tiling of Z are in bijection with spanning subgraphs of G consisting of $m - k$ arcs. In particular, the tiles correspond one-to-one to spanning trees of G.

Proof sketch: Based on (Kavitha et al., 2009): A subset $S \subseteq A$ is spanning if and only if the submatrix of Γ on the columns not in S has rank $m - |S|$.

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Corollary

Let $\mathcal{S}$ denote the set of spanning trees of G. Then

$$\mathrm{vol}(Z) = \sum_{S \in \mathcal{S}} \prod_{a \in A \setminus S} \frac{u_a - \ell_a}{T}.$$

Proof sketch: Every zonotope has a fine zonotopal tiling, each tile corresponds to a spanning tree, the volume of a parallelotope is the absolute value of a determinant of a $\mu \times \mu$ invertible submatrix of Γ', any $\mu \times \mu$-minor of Γ is in $\{-1, 0, 1\}$.

Relation to Timetabling Torus Polytropes

By a volume argument, and since $\mathcal{B}$ is an integral cycle basis, we can show:

Theorem

Each tile of a fine zonotopal tiling of Z contains at most one lattice point. In particular, the number of lattice points in Z is at most the number of spanning trees of G.

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We think of this as a kind of duality: Certain 0-dimensional objects (lattice points) correspond to top-dimensional objects (polytropes).

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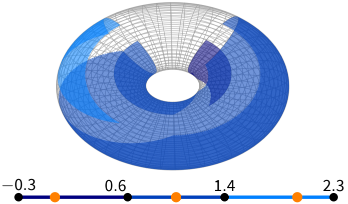
Corollary

There are at most as many non-empty polytropes in $\mathcal{T}$ as there are spanning trees in G.



Cube Faces and Zonotope Tiles

Question

Given a fine zonotopal tiling of Z, consider a tile labeled by a spanning tree S with a lattice point z. Is there a relation between S and $\mathbf{R}(z)$?

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Structures and Faces of X_{LP}

A *structure* is a triple (S, L, U) of subsets of A such that $L \cup U = S$ and $L \cap U = \emptyset$. The structure (S, L, U) is a combinatorial encoding of the face

$$F_{L,U} := \{x \in X_{\mathrm{LP}} \mid x_a = \ell_a \text{ for all } a \in L \text{ and } x_a = u_a \text{ for all } a \in U\}$$

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Lemma

Fix a fine zonotopal tiling of Z. Then for any cell C defined by some spanning subgraph S, there is a structure (S, L, U) such that $C = \frac{1}{T}\Gamma(F_{L,U})$.

Proof sketch: It is non-trivial to check that the translation vectors match. This follows from a formula given by (Richter-Gebert, Ziegler, 1994).

From Tiles to Polytrope Vertices

Theorem

Given a fine zonotopal tiling of Z, let C be a tile defined by a spanning tree S and given by the image of $F_{L,U}$. If C contains a lattice point z, then $\mathbf{R}(z)$ contains a vertex defined by the spanning subgraph in $\overline{G}$ defined by (S, L, U).

We see this as duality as well: A top-dimensional object (tile) gives rise to a 0-dimensional object (polytrope vertex).

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We see this as duality as well: A top-dimensional object (tile) gives rise to a 0-dimensional object (polytrope vertex).

Question

Can we construct a zonotopal tiling by picking vertices from each non-empty polytrope $\mathbf{R}(z)$ in some compatible way?

Construction: From Tropical Vertices to Tiles

For a vertex $i \in V$, define the polyhedron $P_i := X_{\text{LP}} + \text{pos}(\widehat{B_i^\top})$.

Theorem

(1) *Every bounded face of P_i is a face $F_{L,U}$ of X_{LP}.*

(2) *Every bounded face $F_{L,U}$ of P_i corresponds to a structure (S, L, U) such that S is spanning. In particular,* $\dim F_{L,U} \leq \mu$.

(3) *The bounded faces of P_i correspond bijectively to arborescences in $\overline{G}$ rooted at i.*

(4) $\frac{1}{T}\Gamma$ *maps the bounded faces of P_i to the tiles of a fine zonotopal tiling of Z.*

Remarks on the proof: (1) is straightforward, (2) and (3) use network flows, and (4) is a volume argument.

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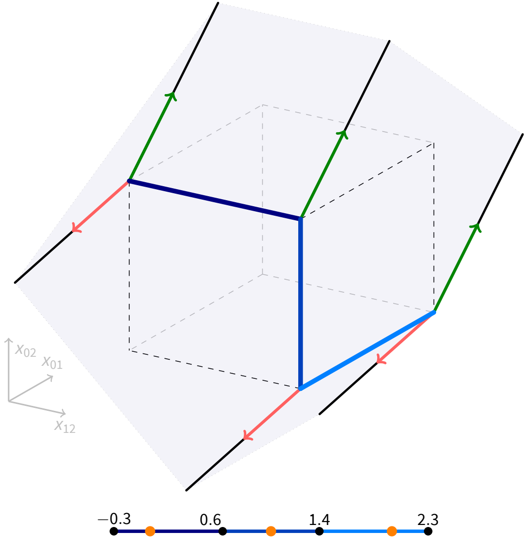
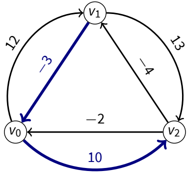
Corollary

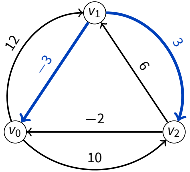
Let $i \in V$. Then there is a fine zonotopal tiling of Z such that any tile containing an integer point z corresponds to a spanning tree structure defined by the i-th tropical vertex of $\mathbf{R}(z)$.

Pictures of the Construction

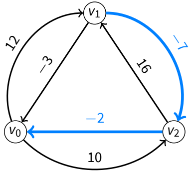




$z = 0$,
$p = (0, 0, 0)$,
$\pi = (0, 3, 10)$



$z = 1$,
$p = (0, 0, 1)$,
$\pi = (0, 3, 6)$



$z = 2$,
$p = (0, 0, 2)$,
$\pi = (0, 9, 2)$

Minimum Width Integral Cycle Bases

The *width* of an integral cycle basis $\mathcal{B}$ is (Liebchen, Peeters, 2009)

$$W_{\mathcal{B}} := \prod_{\gamma \in \mathcal{B}} \left(\left\lfloor \frac{\gamma_+^\top u - \gamma_-^\top \ell}{T} \right\rfloor - \left\lceil \frac{\gamma_+^\top \ell - \gamma_-^\top u}{T} \right\rceil + 1 \right).$$

- By construction, $W_{\mathcal{B}}$ is an upper bound on the number of lattice points in Z.
- $W_{\mathcal{B}}$ hence an upper bound on the number of leaves of a branch-and-bound tree.
- Finding an integral cycle basis of minimum width is hence desirable, but the complexity status is open.
- In practice, this is currently (heuristically) done by a minimum weight undirected cycle basis algorithm (Horton, 1987, de Pina, 1995).

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Lemma

$W_{\mathcal{B}}$ is the number of lattice points in the smallest hyperrectangle containing the cycle offset zonotope Z.

In our two examples, $W_{\mathcal{B}} = 3$ resp. $W_{\mathcal{B}} = 12$. In both cases, $W_{\mathcal{B}}$ equals the number of spanning trees. Coincidence?

Cycle Bases, Spanning Trees, Approximating Width

Lemma (Zonotope volume vs. volume of smallest containing hyperrectangle)

Let $d \in \mathbb{R}^A_{\geq 0}$. *Then*

$$\sum_{S \in \mathcal{S}} \left(\prod_{a \in A \setminus S} d_a \right) \leq \prod_{\gamma \in \mathcal{B}} \left(\sum_{a \in A : \gamma_a \neq 0} d_a \right).$$

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Corollary

For any graph G, the number $|\mathcal{S}|$ *of spanning trees is at most the product of the lengths of the cycles in an integral cycle basis* $\mathcal{B}$ *of G.*

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Corollary

Suppose that $W_{\mathcal{B}} \geq 1$*. Then*

$$|\mathcal{S}| \cdot \left(\frac{\varepsilon}{T} \right)^{\mu} \leq \mathrm{vol}(Z) \leq \prod_{\gamma \in \mathcal{B}} s_\gamma \leq W_{\mathcal{B}} \cdot \prod_{\gamma \in \mathcal{B}} \frac{s_\gamma}{\max\{\lfloor s_\gamma \rfloor, 1\}} < W_{\mathcal{B}} \cdot 2^{\mu},$$

where $\varepsilon := \min\{u_a - \ell_a \mid a \in A\}$ *and* $s_\gamma := \sum_{a \in A: \gamma_a \neq 0} \frac{u_a - \ell_a}{T}$.

The Final Slide

Conclusion

- We connected the problem of periodic timetabling in public transport in a twofold way to discrete geometry: to polytropes and to zonotopes.
- There are close relationships between the polytropes and the zonotopes.
- Details: `arXiv:2204.13501`

The Final Slide

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Outlook

- We have implemented tropical neighborhood search (writing in progress).
- We are also working on a branch-and-bound heuristic for PESP using branching on tropical sectors.
- The cycle offset zonotope seems to be related to Benders decomposition.
- Big question: Can we turn the geometric insights into useful optimization techniques?

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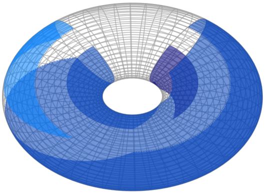
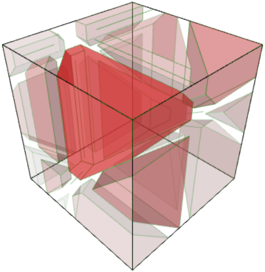


On the tropical and zonotopal geometry of periodic timetabling

Enrico Bortoletto, Niels Lindner, Berenike Masing
Zuse Institute Berlin



Research Seminar on Discrete and Convex Geometry @ TU Berlin

May 18, 2022