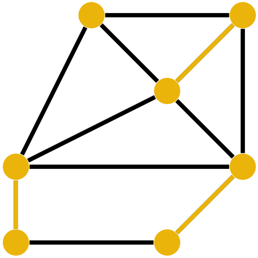
11 Bipartite Matchings

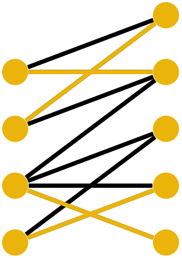
11.1 Definition and Computation

Matchings

Definition 11.1 A matching in an undirected graph $G = (V, E)$ is a subset of edges $M \subseteq E$ with $e \cap e' = \emptyset$ for all $e, e' \in M$ with $e \neq e'$.



Matching M

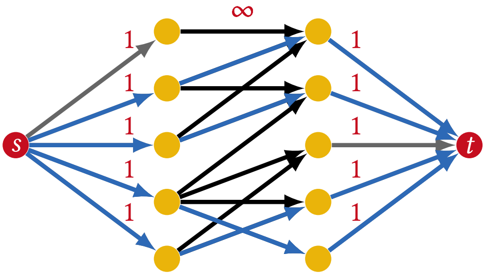


Matching M in a bipartite graph

- recall that a graph $G = (V, E)$ is bipartite if V can be partitioned into L and R such that $E \subseteq \{\{l, r\} : l \in L, r \in R\}$

Maximum Cardinality Matchings

- a matching M is called a maximum matching if it has maximum cardinality, i.e., $|M| \geq |M'|$ for all matchings M'
- a matching is called perfect if $|M| = |V|/2$
- computation of maximum matchings in general graphs $\longrightarrow$ ADM II
- the computation of maximum matchings in bipartite graphs can be done via a reduction to the maximum flow problem in $O(nm)$

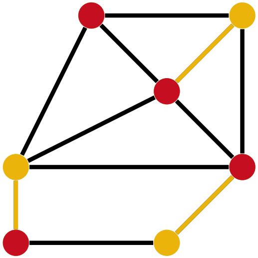


11 Bipartite Matchings

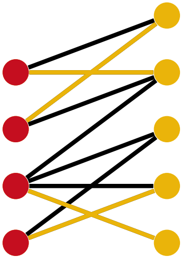
11.2 Kőnig's Theorem

Node Cover

Definition 11.2 A node cover for an undirected graph $G = (V, E)$ is a subset of nodes $C \subseteq V$ with $e \cap C \neq \emptyset$ for all $e \in E$.



Node cover C

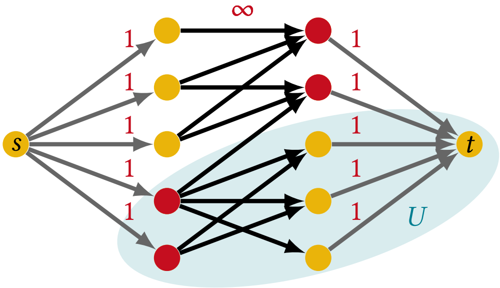


Node cover C in a bipartite graph

Observation: $|M| \leq |C|$ for any matching M and node cover C.

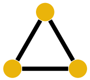
Theorem 11.3 In bipartite graphs, the maximum cardinality of a matching equals the minimum cardinality of a node cover.

Proof: (Idea) Use max-flow min-cut theorem. Consider min-cut $\delta^{-}(U)$:



□

Observation: Kőnig's Theorem does not hold for arbitrary graph:



That is, there can be a 'duality gap' up to a factor of 2:

$|M| \leq |C| \leq 2|M|$ for max matching M and min node cover C. (Why?)

Introduction to

Linear and Combinatorial Optimization

11 Bipartite Matchings

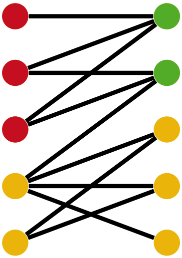
11.3 Hall's Theorem

Perfect Matchings

When does a bipartite graph have a perfect matching?

- $|L| = |R|$ is clearly necessary
- there is no perfect matching if there is a set $S \subseteq L$ with too few potential matches $N(S)$, i.e., $|S| > |N(S)|$ where

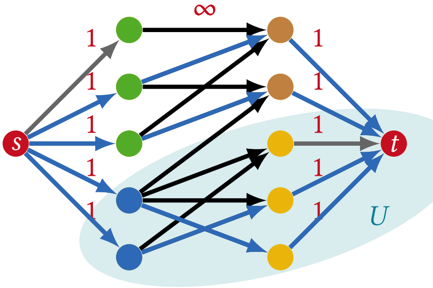
$$N(S) = \{r \in R \,:\, \exists\{l, r\} \in E \text{ with } l \in S\}$$



Hall's Theorem

Theorem 11.4 (Marriage Theorem) A bipartite graph with $|L| = |R|$ has a perfect matching if and only if $|N(S)| \geq |S|$ for all $S \subseteq L$.

Proof:



- if $|N(S)| < |S|$ for some $S \subseteq L$, there is no perfect matching
- suppose no perfect matching
- consider min-cut $\delta^-(U)$, $u(\delta^-(U)) < |L|$
- $L^- = L \setminus U$, $L^+ = L \cap U$, $R^- = R \setminus U$
- $u(\delta^-(U)) = |L^+| + |R^-|$
- $N(L^-) \subseteq R^-$ since $\delta^-(U)$ cannot contain ∞-arcs, thus,

$$|N(L^-)| \leq |R^-| = u(\delta^-(U)) - |L^+| < |L| - |L^+| = |L^-|$$

- choose $S = L^-$

$\square$

11 Bipartite Matchings

11.4 Stable Matchings

Stable Marriage/Matching Problem

Given: set M of men and set W of women with $|M| = |W| = n$
every $m \in M$ has a total preference order over W
every $w \in W$ has a total preference order over M

Task: find a *stable* matching

Stability: no incentive for a pair to undermine assignment by joint action.

- unmatched pair m, w is a blocking pair if man m and woman w prefer each other to their current partners
- Stable matching: perfect matching with no blocking pair.

M	1st	2nd	3rd
Xaver	Anne	Birte	Clara
Yann	Birte	Anne	Clara
Zoltan	Anne	Birte	Clara

W	1st	2nd	3rd
Anne	Yann	Xaver	Zoltan
Birte	Xaver	Yann	Zoltan
Clara	Xaver	Yann	Zoltan

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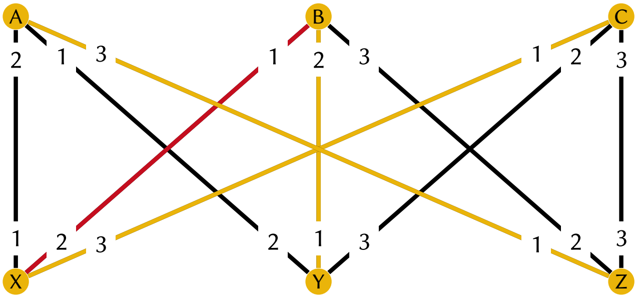
W	1st	2nd	3rd
Anne	Yann	Xaver	Zoltan
Birte	Xaver	Yann	Zoltan
Clara	Xaver	Yann	Zoltan

Example

M	1st	2nd	3rd
Xaver	Anne	Birte	Clara
Yann	Birte	Anne	Clara
Zoltan	Anne	Birte	Clara

W	1st	2nd	3rd
Anne	Yann	Xaver	Zoltan
Birte	Xaver	Yann	Zoltan
Clara	Xaver	Yann	Zoltan



Example

M	1st	2nd	3rd
Xaver	Anne	Birte	Clara
Yann	Birte	Anne	Clara
Zoltan	Anne	Birte	Clara

W	1st	2nd	3rd
Anne	Yann	Xaver	Zoltan
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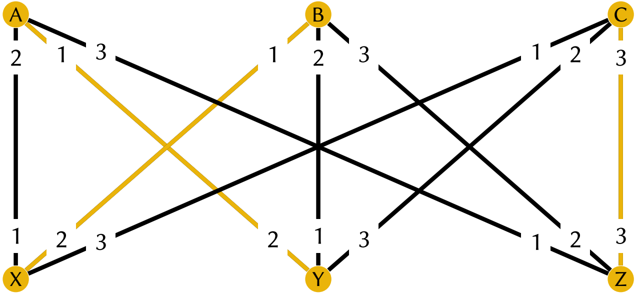

Example

M	1st	2nd	3rd
Xaver	Anne	Birte	Clara
Yann	Birte	Anne	Clara
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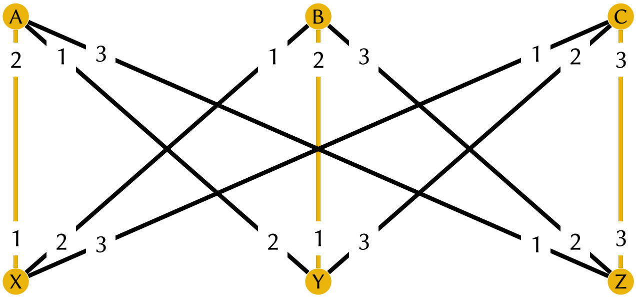


Computing a Stable Matching

- for $m \in M$ and $w, w' \in W$ we write $w \prec_m w'$ if m prefers w to w'
- For $w \in W$ and $m, m' \in M$ we write $m \prec_w m'$ if w prefers m to m'

Gale-Shapley Algorithm

```
initially, all men and women are free (i.e., not engaged)
while there exists a free man m ∈ M
    let w be first woman (w.r.t. ≺_m) to whom m has not yet proposed
    if w is free or engaged to m′ ∈ M with m ≺_w m′
        m and w become engaged (and m′ becomes free)
```

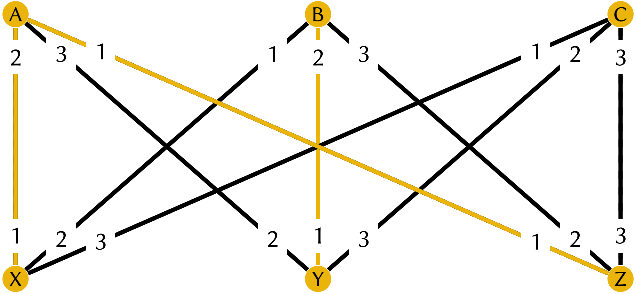
Theorem 11.5 The Gale-Shapley Algorithm finds a stable matching in time $O(n^2)$.

D. Gale, L. S. Shapley: College Admissions and the Stability of Marriage, *American Mathematical Monthly* 69, 9-14, 1962

Gale-Shapley Algorithm: Example

M	1st	2nd	3rd
Xaver	Anne	Birte	Clara
Yann	Birte	Anne	Clara
Zoltan	Anne	Birte	Clara

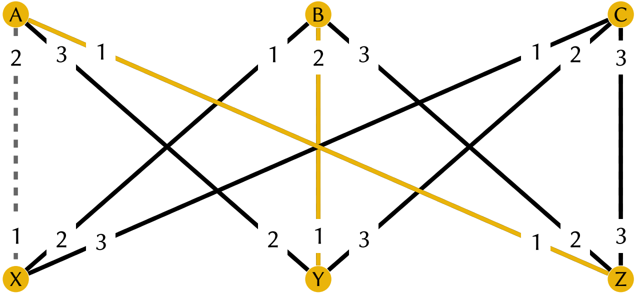
W	1st	2nd	3rd
Anne	Zoltan	Xaver	Yann
Birte	Xaver	Yann	Zoltan
Clara	Xaver	Yann	Zoltan



Gale-Shapley Algorithm: Example

M	1st	2nd	3rd
Xaver	Anne	Birte	Clara
Yann	Birte	Anne	Clara
Zoltan	Anne	Birte	Clara

W	1st	2nd	3rd
Anne	Zoltan	Xaver	Yann
Birte	Xaver	Yann	Zoltan
Clara	Xaver	Yann	Zoltan



Gale-Shapley Algorithm: Example

M	1st	2nd	3rd
Xaver	Anne	Birte	Clara
Yann	Birte	Anne	Clara
Zoltan	Anne	Birte	Clara

W	1st	2nd	3rd
Anne	Zoltan	Xaver	Yann
Birte	Xaver	Yann	Zoltan
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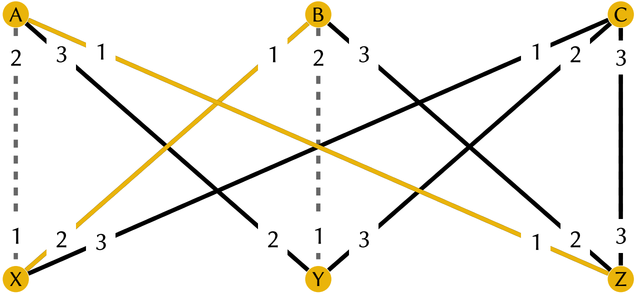

Gale-Shapley Algorithm: Example

M	1st	2nd	3rd
Xaver	Anne	Birte	Clara
Yann	Birte	Anne	Clara
Zoltan	Anne	Birte	Clara

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Anne	Zoltan	Xaver	Yann
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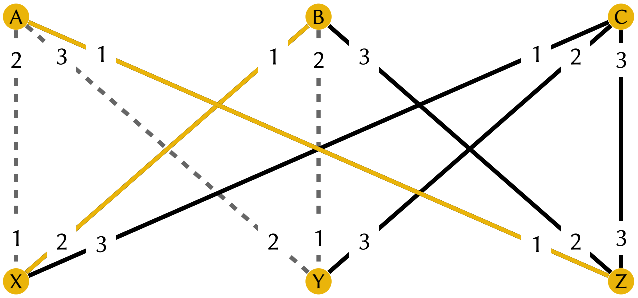

Gale-Shapley Algorithm: Example

M	1st	2nd	3rd
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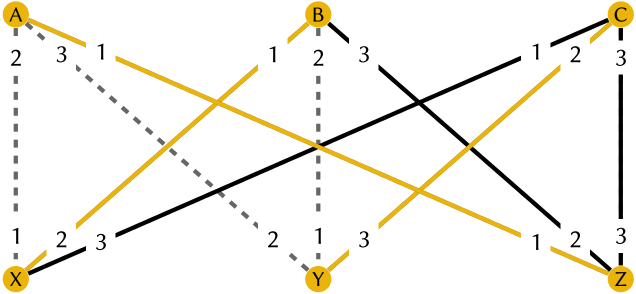
Yann	Birte	Anne	Clara
Zoltan	Anne	Birte	Clara

W	1st	2nd	3rd
Anne	Zoltan	Xaver	Yann
Birte	Xaver	Yann	Zoltan
Clara	Xaver	Yann	Zoltan



Proof of Theorem 11.5

The algorithm computes a perfect matching in time $O(n^2)$ because:

i The edges joining currently engaged couples always form a matching.

ii By construction, once a woman w is engaged, she always remains engaged while her fiancé may change (and only get better w.r.t. $\prec_w$).

iii The woman w in Step 3 always exists: Since $|M| = |W|$ and m is free, there exists a free woman w' to whom m has not yet proposed by ii.

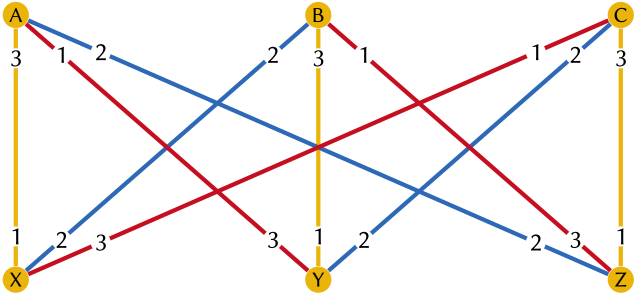
iv In every iteration, a man proposes. Since every man proposes to every woman at most once, the number of iterations is bounded by $O(n^2)$.

We now argue that the computed matching is stable:

- Assume by contradiction that there is an instability m, w, i.e., m is matched with $w' \succ_m w$ and w is matched with $m' \succ_w m$.
- Before proposing to his wife w', m must have proposed to w.
- From then on, by ii, w is always engaged to some man whom she likes at least as good as m. ↯ □

Multiplicity of Stable Matchings

Example:



- All three matchings (yellow, blue, red) are stable.
- The yellow matching found by the Gale-Shapley Algorithm is best possible for all men and worst possible for all women.
- The red matching is best possible for all women and worst possible for all men (found by Gale-Shapley if women propose instead of men).
- The blue matching lies inbetween and might be a good compromise.

Proposing Pays Off!

For $x \in M \cup W$, let best(x) and worst(x) be the best and worst partner (w.r.t. $\prec_x$), respectively, that x can have in any stable matching.

Theorem 11.6 The Gale-Shapley Algorithm matches every man m to best(m) and every woman w to worst(w). In particular, it finds a unique stable matching.

Remarks

- stable matchings can be generalized to arbitrary bipartite graphs (not necessarily complete, $|M| \neq |W|$ etc.)
- in non-bipartite graphs, the problem is known as the Stable Roommates Problem.
- in 2012, the Nobel Memorial Prize in Economics was awarded to Lloyd S. Shapley and Alvin E. Roth "for the theory of stable allocations and the practice of market design."

Proof of Theorem 11.6

Claim: For each $m \in M$, $\text{worst}(\text{best}(m)) = m$.

Proof: Otherwise, for $w := \text{best}(m)$ we get $m \prec_w \text{worst}(w)$.

Consider stable matching where w is matched with $\text{worst}(w)$.

Then, m is matched with some $w' \neq w = \text{best}(m)$ such that $w \prec_m w'$.

As $m \prec_w \text{worst}(w)$ and $w \prec_m w'$, pair m, w is a blocking pair. ⚡ □

Proof of Theorem 11.6:

By contradiction, consider first iteration in which some $m \in M$ is rejected by $w := \text{best}(m)$ in favor of $m' \prec_w m$.

Then, m' has not previously been rejected by $\text{best}(m')$ and thus likes w better than any $w' \neq w$ he can be matched with in a stable matching.

Consider a stable matching where m is matched with $w = \text{best}(m)$.

Then m' is matched with $w' \succ_{m'} w$ and m', w is a blocking pair. ⚡ □