Introduction to

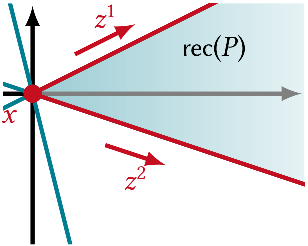
Linear and Combinatorial Optimization

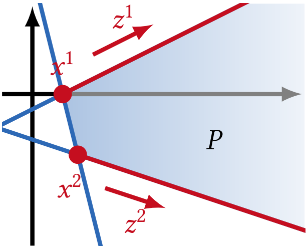
14 Representation of Polyhedra

14.1 Extreme Rays

Extreme Rays

- a one-dimensional face F of a polyhedron is an extreme ray (face) if F has one vertex, i.e., $F = x + \text{cone}(\{z\})$ with $x \in \mathbb{R}^n$, $z \in \mathbb{R}^n \setminus \{0\}$
- we call z an extreme ray
- for a pointed polyhedral cone C, the extreme rays are the points where $n - 1$ linearly independent inequalities are active
- for a pointed polyhedron, the extreme rays are the extreme rays of the recession cone
    - extreme rays of the polyhedron are in the recession cone by Lemma 3.15
    - $n - 1$ linearly independent inequalities $a_i^\top(x + \lambda z) \geq b_i$ active for all $\lambda \geq 0$
    - $n - 1$ linearly independent inequalities $a_i^\top z \geq 0$ active for z





Characterization of Unbounded LPs

Theorem 14.1 Let $C := \{x \in \mathbb{R}^n \mid a_i^\top \cdot x \geq 0,\ i = 1, \dots, m\}$ be a pointed polyhedral cone and $c \in \mathbb{R}^n$. The minimal cost $c^\top \cdot x$ subject to $x \in C$ is equal to $-\infty$ if and only if there is an extreme ray d of C with $c^\top \cdot d < 0$.

Proof: "$\Longleftarrow$" is clear by definition of rays.

"$\Longrightarrow$": Suppose that $\min\{c^\top \cdot x \mid x \in C\}$ is unbounded

- there is $x \in C :\ \ c^\top \cdot x < 0$
- there is $x \in C :\ \ c^\top \cdot x = -1$
- $P := \{x \in \mathbb{R}^n \mid c^\top \cdot x = -1,\ a_i^\top \cdot x \geq 0,\ i = 1, \dots, m\} \neq \emptyset$
- since C is pointed (i.e., $a_1, \dots, a_m$ span $\mathbb{R}^n$), P is pointed as well
- consider extreme point $d \in P$
- there are n linearly independent constraints active at d
- there are $n - 1$ linearly independent constraints $a_i^\top \cdot x \geq 0$ active at d
- d is an extreme ray of C (note that $d \neq 0$ since $c^\top \cdot d = -1$) $\square$

Characterization of Unbounded LPs (Cont.)

- Theorem 14.1 also holds for pointed polyhedra:

Theorem 14.2 Let $P \subseteq \mathbb{R}^n$ be a pointed polyhedron and $c \in \mathbb{R}^n$. The minimal cost $c^\top \cdot x$ subject to $x \in P$ is equal to $-\infty$ if and only if there is an extreme ray d of P with $c^\top \cdot d < 0$.

- if the simplex method observes that an LP is unbounded, the corresponding jth basic direction is an extreme ray d with $c^\top \cdot d < 0$

Proof of Theorem 14.2

Let $P = \{x \in \mathbb{R}^n \mid A \cdot x \geq b\}$ be pointed.

"$\Longleftarrow$" is clear by definition of rays.

"$\Longrightarrow$": Consider infeasible dual LP:

$$\max\ p^\top \cdot b \qquad \text{s.t.} \quad p^\top \cdot A = c^\top, \quad p \geq 0$$

- replace objective function by $p^\top \cdot 0 \Longrightarrow$ problem remains infeasible:

$$\max\ p^\top \cdot 0 \qquad \text{s.t.} \quad p^\top \cdot A = c^\top, \quad p \geq 0$$

- corresponding primal LP is feasible and thus unbounded:

$$\min\ c^\top \cdot x \qquad \text{s.t.} \quad A \cdot x \geq 0$$

- by Theorem 14.1, there is an extreme ray d of $\{x \mid A \cdot x \geq 0\}$ with $c^\top \cdot d < 0$.
- since $\{x \mid A \cdot x \geq 0\}$ is recession cone of P, d is extreme ray of P. □

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14.2 Resolution Theorem

Resolution Theorem

- call a set $W = \{w^1, \ldots, w^r\}$ of extreme rays complete if $\lambda w \in W$ for all extreme rays w for some $\lambda > 0$

Theorem 14.3 Let $P := \{x \in \mathbb{R}^n \mid A \cdot x \geq b\} \neq \emptyset$ be pointed. Let $x^1, \ldots, x^k$ be the extreme points and $w^1, \ldots, w^r$ a complete set of extreme rays of P. Then,

$$P = \left\{ \sum_{i=1}^{k} \lambda_i \cdot x^i + \sum_{j=1}^{r} \theta_j \cdot w^j \;\middle|\; \lambda_i, \theta_j \geq 0,\ \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

Corollary 14.4 A non-empty polytope is equal to the convex hull of its extreme points. □

Corollary 14.5 Every element of a pointed polyhedral cone is a non-negative linear combination (i.e., a conic combination) of its extreme rays. □

Proof of Theorem 14.3

Let $Q := \left\{ \sum_{i=1}^{k} \lambda_i \cdot x^i + \sum_{j=1}^{r} \theta_j \cdot w^j \mid \lambda_i, \theta_j \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}$.

$P \supseteq Q$: Clear by convexity of P and by definition of rays of P.

$P \subseteq Q$: Assume by contradiction that there is a $z \in P \setminus Q$.

Since $z \notin Q$, the following LP is infeasible:

$$\begin{array}{ll} \max & \sum_{i=1}^{k} 0 \cdot \lambda_i + \sum_{j=1}^{r} 0 \cdot \theta_j \\ \text{s.t.} & \sum_{i=1}^{k} \lambda_i \cdot x^i + \sum_{j=1}^{r} \theta_j \cdot w^j = z, \quad \sum_{i=1}^{k} \lambda_i = 1 \quad \lambda, \theta \geq 0 \end{array}$$

The corresponding dual LP is feasible and thus unbounded:

$$\min \; p^\top \cdot z + q \qquad \text{s.t.} \quad p^\top \cdot x^i + q \geq 0 \quad \forall i, \qquad p^\top \cdot w^j \geq 0 \quad \forall j$$

There is a solution $(\bar{p}, \bar{q})$ with $\bar{p}^\top \cdot z + \bar{q} < 0$ and thus

$$\bar{p}^\top \cdot z < \bar{p}^\top \cdot x^i \quad \forall i \qquad \text{and} \qquad \bar{p}^\top \cdot w^j \geq 0 \quad \forall j \tag{$\star$}$$

Proof of Theorem 14.3 (Cont.)

For this fixed vector $\bar{p}$, consider the LP:

$$\min\ \bar{p}^\top \cdot x \qquad \text{s.t.} \quad A \cdot x \geq b$$

Notice that z is a feasible solution to this LP.

Case 1: The LP has finite optimal cost.

Then, there is an optimal extreme point x^i for some i.

In particular, $\bar{p}^\top \cdot z \geq \bar{p}^\top \cdot x^i$ for this i, a contradiction to ($\star$).

Case 2: The LP is unbounded.

By Theorem 14.2, there is an extreme ray w^j with $\bar{p}^\top \cdot w^j < 0$,

again a contradiction to ($\star$). □

Converse to the Resolution Theorem

Definition 14.6 A set $Q \subseteq \mathbb{R}^n$ is finitely generated if there are $x^1, \dots, x^k, w^1, \dots, w^r \in \mathbb{R}^n$ such that

$$Q = \left\{ \sum_{i=1}^{k} \lambda_i \cdot x^i + \sum_{j=1}^{r} \theta_j \cdot w^j \mid \lambda_i, \theta_j \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

Remark

- the Resolution Theorem states that a polyhedron with at least one extreme point is finitely generated
- this is also true for general polyhedra

Theorem 14.7 A finitely generated set Q is a polyhedron. In particular, the convex hull of finitely many vectors is a polytope.

Proof of Theorem 14.7

For some $z \in \mathbb{R}^n$, consider the LP

$$\begin{aligned} \max \quad & \textstyle\sum_{i=1}^{k} 0 \cdot \lambda_i + \sum_{j=1}^{r} 0 \cdot \theta_j \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^{k} \lambda_i \cdot x^i + \sum_{j=1}^{r} \theta_j \cdot w^j = z, \quad \sum_{i=1}^{k} \lambda_i = 1 \quad \lambda, \theta \geq 0 \end{aligned}$$

Then, $z \in Q$ if and only if the LP is feasible and bounded.

Thus, $z \in Q$ if and only if the dual LP is bounded:

$$\min \; p^\top \cdot z + q \qquad \text{s.t.} \quad p^\top \cdot x^i + q \geq 0 \quad \forall i, \qquad p^\top \cdot w^j \geq 0 \quad \forall j$$

Convert the dual LP to standard form:

$$\begin{aligned} \min \quad & (p^+ - p^-)^\top \cdot z + (q^+ - q^-) & \\ \text{s.t.} \quad & (p^+ - p^-)^\top \cdot x^i + (q^+ - q^-) - \alpha_i = 0 & \forall i \\ & (p^+ - p^-)^\top \cdot w^j - \beta_j = 0 & \forall j \\ & p^+, p^-, q^+, q^-, \alpha, \beta \geq 0 & \end{aligned}$$

The set of feasible solutions of this LP in standard form is pointed.

Proof of Theorem 14.7 (Cont.)

By Theorem 14.1, the dual LP in standard form is bounded if and only if

$$(p^+ - p^-)^\top \cdot z + (q^+ - q^-) \geq 0 \qquad (\star)$$

holds for all, finitely many, extreme rays $(p^+, p^-, q^+, q^-, \alpha, \beta)$.

Conclusion:

$$\begin{aligned} z \in Q \quad &\Longleftrightarrow \quad \text{dual LP is bounded} \\ &\Longleftrightarrow \quad z \text{ fulfills finitely many linear inequalities } (\star) \end{aligned}$$

Thus, Q is a polyhedron. □

Representation of Polyhedra

Conclusion: There are two ways of representing a polyhedron:

i in terms of a finite set of linear constraints (outer representation);

ii as a finitely generated set, in terms of its extreme points and rays (inner representation).

Remarks

- Passing from one type of description to the other is, in general, a complicated computational task.
- One description can be small while the other one is huge. Examples:
  - An n-dimensional cube is given by $2n$ linear constraints and has 2^n extreme points.
  - A representation of the convex hull of the $2n$ points

$$e_1, -e_1, e_2, -e_2, \dots, e_n, -e_n$$

in terms of linear constraints needs at least 2^n linear inequalities.