16 Sensitivity Analysis for LPs

16.1 Local Sensitivity Analysis

Local Sensitivity Analysis

Consider a primal-dual pair of linear programs:

$$\begin{aligned} \min \quad & c^\top \cdot x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \max \quad & p^\top \cdot b \\ \text{s.t.} \quad & p^\top \cdot A \leq c^\top \end{aligned}$$

Let B be an optimal basis for the primal LP, i.e.,

$$A_B^{-1} \cdot b \geq 0 \qquad \text{(feasibility)},$$

$$c^\top - c_B^\top \cdot A_B^{-1} \cdot A \geq 0 \qquad \text{(optimality)},$$

and let x^* be a corresponding optimal basic solution.

Questions

- Under what conditions does B remain feasible and optimal when problem data is being changed?
- What if B is no longer feasible or optimal after the change?

Adding a New Variable

Add a new variable x_{n+1} to the primal LP

$$\begin{aligned} \min\ & c^\top \cdot x + c_{n+1} \cdot x_{n+1} \\ \text{s.t.}\ & A \cdot x + A_{n+1} \cdot x_{n+1} = b \\ & (x, x_{n+1}) \geq 0 \end{aligned}$$

- $(x^*, 0)$ is a basic feasible solution to the new problem.
- B remains optimal if $\bar{c}_{n+1} := c_{n+1} - c_B^\top \cdot A_B^{-1} \cdot A_{n+1} \geq 0$.
- Otherwise apply the primal simplex algorithm to reoptimize!

Adding a New Inequality

Add a new inequality $a_{m+1}^\top \cdot x \geq b_{m+1}$ to the primal LP

- if x^* satisfies the new constraint, it remains optimal
- otherwise, introduce slack variable $x_{n+1} \geq 0$ and rewrite:

$$a_{m+1}^\top \cdot x - x_{n+1} = b_{m+1}$$

- new matrix $\bar{A} = \begin{pmatrix} A & 0 \\ a_{m+1}^\top & -1 \end{pmatrix}$, new basis $\bar{B} = (B(1), \dots, B(m), n+1)$
- $\bar{A}_{\bar{B}} = \begin{pmatrix} A_B & 0 \\ \tilde{a}^\top & -1 \end{pmatrix}$ with $\tilde{a} = (a_{m+1,B(1)}, \dots, a_{m+1,B(m)})^\top$
- $\bar{A}_{\bar{B}}^{-1} = \begin{pmatrix} A_B^{-1} & 0 \\ \tilde{a}^\top A_B^{-1} & -1 \end{pmatrix}$ with corr. basic solution $(x^*, \underbrace{a_{m+1}^\top \cdot x^* - b_{m+1}}_{<0})$
- new reduced costs $\left[c^\top, 0\right] - \left[c_B^\top, 0\right] \bar{A}_{\bar{B}}^{-1} \cdot \bar{A} = \left[c^\top - c_B^\top \cdot A_B^{-1} \cdot A, 0\right] \geq 0$
- apply the dual simplex algorithm to reoptimize!

Adding a New Equation

Add a new equation $a_{m+1}^\top \cdot x = b_{m+1}$ to the primal LP.

- if x^* satisfies the new constraint, it remains optimal
- otherwise, w.l.o.g. $a_{m+1}^\top \cdot x^* > b_{m+1}$; consider auxiliary problem (with M large enough):

$$\begin{aligned} \min\ & c^\top \cdot x + M \cdot x_{n+1} \\ \text{s.t.}\ & A \cdot x = b \\ & a_{m+1}^\top \cdot x - x_{n+1} = b_{m+1} \\ & (x, x_{n+1}) \geq 0 \end{aligned}$$

- new feasible basis $A_{\bar{B}} := \begin{pmatrix} B & 0 \\ \tilde{a}^\top & -1 \end{pmatrix}$, $A_{\bar{B}}^{-1} = \begin{pmatrix} A_B^{-1} & 0 \\ \tilde{a}^\top \cdot A_B^{-1} & -1 \end{pmatrix}$, and associated basic feasible solution $(x^*, \underbrace{a_{m+1}^\top \cdot x^* - b_{m+1}}_{>0})$
- apply the primal simplex algorithm to reoptimize!

Changing the Right-Hand Side

Change b to $b + \delta \cdot e_i$, that is, only b_i is changed to $b_i + \delta$

- Optimality condition $\bar{c} \geq 0$ is not affected!
- Feasibility: $A_B^{-1} \cdot (b + \delta \cdot e_i) \geq 0$?
- Let $g := (\beta_{1i}, \ldots, \beta_{mi})^\top$ be the ith column of A_B^{-1}:

$$A_B^{-1} \cdot (b + \delta \cdot e_i) = x_B^* + \delta \cdot g \geq 0$$

$$\iff \quad x_{B(j)}^* + \delta \cdot \beta_{ji} \geq 0 \qquad \text{for } j = 1, \ldots, m$$

$$\iff \quad \max_{j : \beta_{ji} > 0} -\frac{x_{B(j)}^*}{\beta_{ji}} \leq \delta \leq \min_{j : \beta_{ji} < 0} -\frac{x_{B(j)}^*}{\beta_{ji}}$$

- Otherwise, apply the dual simplex algorithm to reoptimize!

Changing the Cost Vector

Change c to $c + \delta \cdot e_j$, that is, only c_j is changed to $c_j + \delta$

- Feasibility is not affected but optimality – apply the primal simplex algorithm to reoptimize!

- Case 1: x_j is non-basic $\implies$ only $\bar{c}_j$ affected:

$$\hat{c}_j := c_j + \delta - c_B^\top \cdot A_B^{-1} \cdot A_j = \bar{c}_j + \delta$$

Thus, B remains optimal, if and only if $\delta \geq -\bar{c}_j$.

- Case 2: $x_j = x_{B(\ell)}$ is basic $\implies$ all reduced costs affected:

$$c_i - (c_B + \delta \cdot e_\ell)^\top \cdot A_B^{-1} \cdot A_i \geq 0 \qquad \text{for all } i \neq j$$

$$\iff \qquad \bar{c}_i - \delta \cdot q_{\ell i} \geq 0 \qquad \text{for all } i \neq j$$

where $q_{\ell i} = \ell$th entry of $A_B^{-1} \cdot A_i$.

Changing a Column of A

Change A_j to $A_j + \delta \cdot e_i$, that is, only a_{ij} is changed to $a_{ij} + \delta$

Case 1: A_j is a non-basic column.

- B is still feasible but $\bar{c}_j$ is affected.
- If $c_j - \underbrace{p^\top}_{c_B^\top \cdot A_B^{-1}} \cdot (A_j + \delta \cdot e_i) = \bar{c}_j - \delta \cdot p_i \geq 0$, then B remains optimal.
- Otherwise, apply the primal simplex algorithm to reoptimize!

Case 2: A_j is a basic column: See exercises.

16 Sensitivity Analysis for LPs

16.2 Global Dependences

$$\begin{aligned} P(b) &:= \{x \mid A \cdot x = b,\ x \geq 0\} \\ S &:= \{b \mid P(b) \neq \emptyset\} = \{A \cdot x \mid x \geq 0\} \qquad \text{(convex)} \\ F(b) &:= \min_{x \in P(b)} c^\top \cdot x \qquad \text{for } b \in S \end{aligned}$$

Assume that the dual feasible set is non-empty: $\{p \mid p^\top \cdot A \leq c^\top\} \neq \emptyset$

$$\Longrightarrow \quad F(b) > -\infty \qquad \text{for all } b \in S.$$

Consider fixed $b^* \in S$ and assume that B is a non-degenerate optimal basis:

$$x_B = A_B^{-1} \cdot b^* > 0 \ , \qquad \bar{c}^\top = c^\top - {c_B}^\top \cdot A_B^{-1} \cdot A \geq 0 \ .$$

Changing b^* to b with $b - b^*$ sufficiently small, $A_B^{-1} \cdot b$ remains non-negative and B is still optimal.

$$\Longrightarrow \quad F(b) = c_B^\top \cdot A_B^{-1} \cdot b = p^\top \cdot b \qquad \text{for } b \text{ close to } b^*.$$

Theorem 16.1 $F(b)$ is a convex function of b. It is linear in vicinity of b^* with gradient p.

Proof of Theorem 16.1

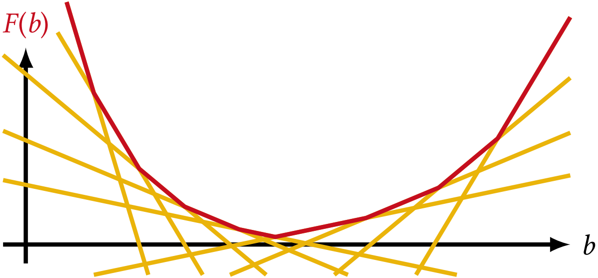
Consider the feasible dual LP:

$$\max \; p^\top b \qquad \text{s.t.} \quad p^\top A \le c^\top.$$

W.l.o.g. $\operatorname{rank}(A) = m$ such that $Q := \left\{ p \in \mathbb{R}^m \mid p^\top A \le c^\top \right\}$ is pointed. Let $p_1, \ldots, p_N$ be the extreme points of Q. By strong duality

$$F(b) = \max_{i=1,\ldots,N} p_i^\top b \qquad \text{for } b \in S.$$

I.e., F is maximum of N linear functions, thus piecewise linear and convex.



Linear in the vicinity of b^* with gradient p: see above. □

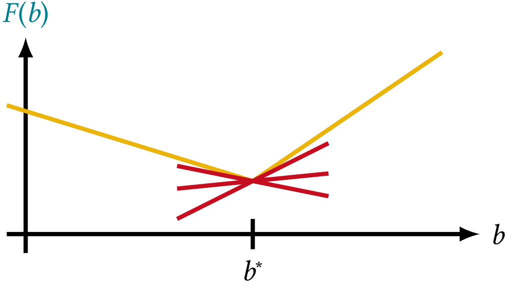
Set of all Dual Optimal Solutions

Definition 16.2 Let $S \subseteq \mathbb{R}^n$ convex, $F : S \longrightarrow \mathbb{R}$ convex, and $b^* \in S$. Then $p \in \mathbb{R}^n$ is a subgradient of F at b^* if

$$F(b^*) + p^\top \cdot (b - b^*) \leq F(b) \qquad \text{for all } b \in S.$$

Remarks:

- If F is differentiable at b^*, then there is a unique subgradient (the gradient $p = \nabla F(b^*)$).
- If b^* is a breakpoint of F, then there are several subgradients.



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Remarks:

- If F is differentiable at b^*, then there is a unique subgradient (the gradient $p = \nabla F(b^*)$).
- If b^* is a breakpoint of F, then there are several subgradients.

Let $F(b) := \min\{c^\top \cdot x \mid A \cdot x = b,\ x \geq 0\}$.

Theorem 16.3 Suppose that the LP $\min c^\top x$, s.t. $Ax = b^*$, $x \geq 0$ is feasible and bounded. Then p is an optimal solution to the dual LP if and only if p is a subgradient of F at b^*.

Proof of Theorem 16.3

"$\Longrightarrow$": Let p an optimal dual solution. Then $p^\top b^* = F(b^*)$ (strong duality). For $b \in S$ it holds that $p^\top b \leq F(b)$ (weak duality).

$$\Longrightarrow \quad F(b^*) + p^\top (b - b^*) = \underbrace{F(b^*) - p^\top b^*}_{=0} + p^\top b \leq F(b)$$

"$\Longleftarrow$": Let p be a subgradient of F at b^*, that is,

$$F(b^*) + p^\top (b - b^*) \leq F(b) \qquad \text{for all } b \in S. \tag{$\star$}$$

Let $x \geq 0$ and $b := Ax$. Then, $x \in P(b)$ and $F(b) \leq c^\top x$. Thus,

$$p^\top Ax = p^\top b \overset{(\star)}{\leq} F(b) - F(b^*) + p^\top b^* \leq c^\top x - F(b^*) + p^\top b^*. \tag{$\star\star$}$$

This inequality holds for all $x \geq 0 \quad \Longrightarrow \quad p^\top A \leq c^\top$ (dual feasibility).

For $x = 0$ inequal. ($\star\star$) yields $F(b^*) \leq p^\top b^*$, i.e., p is optimal dual sol. □

Global Dependence on the Cost Vector

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ with $\{x \mid A \cdot x = b,\ x \geq 0\} \neq \emptyset$ and for $c \in \mathbb{R}^n$

$$\begin{aligned} G(c) &:= \min\{c^\top \cdot x \mid A \cdot x = b,\ x \geq 0\} \\ Q(c) &:= \{p \in \mathbb{R}^m \mid p^\top \cdot A \leq c^\top\} \\ T &:= \{c \in \mathbb{R}^n \mid Q(c) \neq \emptyset\} \qquad \text{(convex).} \end{aligned}$$

Note that $T = \{c \mid G(c) > -\infty\}$ and, for $c \in T$,

$$G(c) = \min_{i=1,\ldots,N} c^\top \cdot x^i \quad \text{where } x^1, \ldots, x^N \text{ are the basic feasible solutions.}$$

Theorem 16.4 Consider a feasible linear program in standard form.

i The set T is convex.

ii $G(c)$ is a concave function on T.

iii If, for some $c \in T$, the primal LP has a unique optimal solution x^*, then G is linear in the vicinity of c and its gradient is equal to x^*. □

Introduction to

Linear and Combinatorial Optimization

16 Sensitivity Analysis for LPs

16.3 Parametric LPs

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c, d \in \mathbb{R}^n$.

Parametric Linear Program.

Solve, for all $\theta \in \mathbb{R}$:

$$\begin{aligned} g(\theta) \;:= \min\; & (c + \theta \cdot d)^\top \cdot x \\ \text{s.t.}\; & A \cdot x = b \\ & x \geq 0 \end{aligned}$$

Assume that $\{x \mid A \cdot x = b,\ x \geq 0\} \neq \emptyset$. Then

$$g(\theta) = \min_{i=1,\ldots,N} (c + \theta \cdot d)^\top \cdot x^i$$

for those θ with $g(\theta) > -\infty$, where $x^1, \ldots, x^N$ are the extreme points.

$$
\begin{array}{rrrrrrrrrl}
g(\theta) & := & \min & (-3+2\theta)\,x_1 & + & (3-\theta)\,x_2 & + & x_3 & & \\
 & & \text{s.t.} & x_1 & + & 2\,x_2 & - & 3\,x_3 & \leq & 5 \\
 & & & 2\,x_1 & + & x_2 & - & 4\,x_3 & \leq & 7 \\
 & & & & & & & x_1,\ x_2,\ x_3 & \geq & 0
\end{array}
$$

Introduce slack variables and set up the simplex tableau:

		x_1	x_2	x_3	x_4	x_5
	0	$-3+2\theta$	$3-\theta$	1	0	0
$x_4 =$	5	1	2	-3	1	0
$x_5 =$	7	2	1	-4	0	1

If $-3+2\theta \geq 0$ and $3-\theta \geq 0$ (i.e., $\theta \in \left[\frac{3}{2}, 3\right]$), this basis is optimal, i.e.,

$$
g(\theta) = 0 \qquad \text{for } \theta \in \left[\frac{3}{2}, 3\right].
$$

Parametric LP Example (Cont.)

If $\theta > 3$, then $\bar{c}_2 < 0$ and we do a pivoting step:

	$-\frac{15}{2} + \frac{5}{2}\theta$	$-\frac{9}{2} + \frac{5}{2}\theta$	0	$\frac{11}{2} - \frac{3}{2}\theta$	$-\frac{3}{2} + \frac{1}{2}\theta$	0
$x_2 =$	$\frac{5}{2}$	$\frac{1}{2}$	1	$-\frac{3}{2}$	$\frac{1}{2}$	0
$x_5 =$	$\frac{9}{2}$	$\frac{3}{2}$	0	$-\frac{5}{2}$	$-\frac{1}{2}$	1

This basic solution is optimal for $3 \leq \theta \leq \frac{11}{3}$, i.e.,

$$g(\theta) = \frac{15}{2} - \frac{5}{2}\theta \qquad \text{for } \theta \in \left[3, \frac{11}{3}\right].$$

If $\theta > \frac{11}{3}$, then $\bar{c}_3 < 0$ and the problem is unbounded, i.e.,

$$g(\theta) = -\infty \qquad \text{for } \theta > \frac{11}{3}.$$

Parametric LP Example (Cont.)

Return to the initial tableau on Slide 16|17:

If $\theta < \frac{3}{2}$, then $\bar{c}_1 < 0$ and we do a pivoting step:

	$\frac{21}{2} - 7\theta$	0	$\frac{9}{2} - 2\theta$	$-5 + 4\theta$	0	$\frac{3}{2} - \theta$
$x_4 =$	$\frac{3}{2}$	0	$\frac{3}{2}$	-1	1	$-\frac{1}{2}$
$x_1 =$	$\frac{7}{2}$	1	$\frac{1}{2}$	-2	0	$\frac{1}{2}$

This basic solution is optimal for $\frac{5}{4} \le \theta \le \frac{3}{2}$, i.e.,

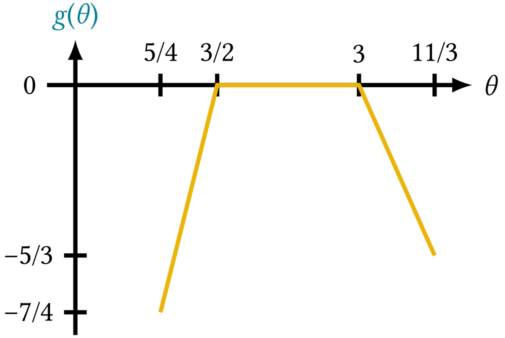
$$g(\theta) = -\frac{21}{2} + 7\theta \qquad \text{for } \theta \in \left[\frac{5}{4}, \frac{3}{2}\right].$$

If $\theta < \frac{5}{4}$, then $\bar{c}_3 < 0$ and the problem is unbounded, i.e.,

$$g(\theta) = -\infty \qquad \text{for } \theta < \frac{5}{4}.$$

Summarizing,

$$g(\theta) = \begin{cases} -\infty & \text{if } \theta < \frac{5}{4} \text{ or } \theta > \frac{11}{3}, \\ -\frac{21}{2} + 7\theta & \text{if } \theta \in \left[\frac{5}{4}, \frac{3}{2}\right], \\ 0 & \text{if } \theta \in \left[\frac{3}{2}, 3\right], \\ \frac{15}{2} - \frac{5}{2}\theta & \text{if } \theta \in \left[3, \frac{11}{3}\right]. \end{cases}$$



Remarks

- The demonstrated approach works for arbitrary parametric LPs
- With an anti-cycling rule (e.g., lexicographic pivoting rule), the method terminates after finitely many iterations. (Cannot visit a basis more than once.)

Bad news:

- Number of iterations can be exponential in the input size.
- Even worse, the function g can have exponentially many breakpoints.
- That is, the parametric LP can have exponentially many different optimal solutions over the entire range of the parameter θ.