2 Linear Programming Basics

2.1 Forms of Linear Programs

Example of a Linear Program

$$\begin{array}{rrcrcrcrl} \text{minimize} & 2x_1 & - & x_2 & + & 4x_3 & & & \\ \text{subject to} & x_1 & + & x_2 & & & + & x_4 & \leq 2 \\ & & & 3x_2 & - & x_3 & & & = 5 \\ & & & & & x_3 & + & x_4 & \geq 3 \\ & x_1 & & & & & & & \geq 0 \\ & & & & & x_3 & & & \leq 0 \end{array}$$

Remarks

- objective function linear in variable vector $x = (x_1, x_2, x_3, x_4)^\top$
- constraints are linear inequalities and linear equations
- in this example, the last two constraints are special:
  non-negativity and non-positivity constraint, respectively

General Linear Program

$$
\begin{aligned}
\text{minimize} \quad & c^\top x \\
\text{subject to} \quad & {a_i}^\top x \geq b_i && \text{for } i \in M_1, \\
& {a_i}^\top x = b_i && \text{for } i \in M_2, \\
& {a_i}^\top x \leq b_i && \text{for } i \in M_3, \\
& x_j \geq 0 && \text{for } j \in N_1, \\
& x_j \leq 0 && \text{for } j \in N_2,
\end{aligned}
$$

with $c \in \mathbb{R}^n$, $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for $i \in M_1 \dot\cup M_2 \dot\cup M_3$ (finite index sets), and $N_1, N_2 \subseteq \{1, \ldots, n\}$ given.

- $x \in \mathbb{R}^n$ satisfying all constraints is a feasible solution
- feasible solution x^* is optimal solution if

$$c^\top x^* \leq c^\top x \qquad \text{for all feasible solutions } x$$

- linear program is infeasible if there exists no feasible solution (feasible set X is empty)
- linear program is unbounded if, for all $k \in \mathbb{R}$, there is a feasible solution $x \in \mathbb{R}^n$ with $c^\top x \leq k$

Special Forms of Linear Programs

- maximizing $c^\top x$ is equivalent to minimizing $-c^\top x$

- any linear program can be written in the form

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax \geq b \end{aligned}$$

for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$:

  - rewrite $a_i{}^\top x = b_i$ as $a_i{}^\top x \geq b_i \;\wedge\; a_i{}^\top x \leq b_i$
  - rewrite $a_i{}^\top x \leq b_i$ as $-a_i^\top x \geq -b_i$
  - rewrite $x_j \geq 0$ as $e_j^\top x \geq 0$
  - rewrite $x_j \leq 0$ as $-e_j^\top x \geq 0$

Reduction to Standard Form

Every linear program can be brought into standard form

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n.$$

i elimination of free (unbounded) variables x_j:
replace x_j with $x_j = x_j^+ - x_j^-$, $x_j^+, x_j^- \geq 0$

ii elimination of non-positive variables x_j:
replace $x_j \leq 0$ with $(-x_j) \geq 0$

iii elimination of inequality constraint $a_i^\top x \leq b_i$:
introduce slack variable $s_i \geq 0$ and rewrite: $a_i^\top \cdot x + s_i = b_i$

iv elimination of inequality constraint $a_i^\top \cdot x \geq b_i$:
introduce slack variable $s_i \geq 0$ and rewrite: $a_i^\top \cdot x - s_i = b_i$

Example

The linear program

$$\begin{array}{rrcrl} \min & 2\,x_1 & + & 4\,x_2 & \\ \text{s.t.} & x_1 & + & x_2 & \geq 3 \\ & 3\,x_1 & + & 2\,x_2 & = 14 \\ & x_1 & & & \geq 0 \end{array}$$

is equivalent to the following standard form problem:

$$\begin{array}{rrcrcrcrl} \min & 2\,x_1 & + & 4\,x_2^+ & - & 4\,x_2^- & & & \\ \text{s.t.} & x_1 & + & x_2^+ & - & x_2^- & - & x_3 & = 3 \\ & 3\,x_1 & + & 2\,x_2^+ & - & 2\,x_2^- & & & = 14 \\ & & & & & & & x_1,\ x_2^+,\ x_2^-,\ x_3 & \geq 0 \end{array}$$

2 Linear Programming Basics

2.2 Examples

Example: Diet Problem

Given:
- n different foods, m different nutrients
- a_{ij} := amount of nutrient i in one unit of food j
- b_i := requirement of nutrient i in some ideal diet
- u_i := upper limit of nutrient i in some ideal diet
- c_j := cost of one unit of food j

Task: find a cheapest ideal diet consisting of foods $1, \dots, n$

Formulation as LP

variables x_j, $j = 1, \dots, n$ with interpretation units of food j in the diet

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax \geq b \\ & Ax \leq u \\ & x \geq 0 \end{aligned}$$

with $A = (a_{ij}) \in \mathbb{R}^{m \times n}$, $b = (b_i) \in \mathbb{R}^m$, $c = (c_j) \in \mathbb{R}^n$.

The Moment Problem in probability

Given:
- $\mu, \sigma, \alpha \in \mathbb{R}$
- a set $S = \{x_1, \dots, x_n\} \subset \mathbb{R}$, a function $f : \mathbb{R} \longrightarrow \mathbb{R}$

Task: find the best possible upper bound for the probability that $f(X) \leq \alpha$, where X is a random variable taking values in S with expected value μ and variance at most σ^2.

Formulation as LP

variables p_i, $i = 1, \dots, n$ with interpretation $\mathbb{P}[X = x_i] = p_i$

$$\begin{aligned}
\max \quad & \sum_{i=1}^{n} p_i \, \chi_{\{i : f(x_i) \leq \alpha\}} \\
\text{s.t.} \quad & \sum_{i=1}^{n} p_i x_i = \mu \\
& \sum_{i=1}^{n} p_i (x_i - \mu)^2 \leq \sigma^2 \\
& \sum_{i=1}^{n} p_i = 1 \\
& p \geq 0.
\end{aligned}$$

Example: LP relaxations

Definition 2.1 Let $X \subseteq Y$ and $f : Y \longrightarrow \mathbb{R}$ and consider the optimization problems

$$\text{minimize} \quad f(x) \qquad \text{subject to} \quad x \in Y, \tag{2.1}$$

$$\text{minimize} \quad f(x) \qquad \text{subject to} \quad x \in X. \tag{2.2}$$

Then, (2.1) is called a relaxation of (2.2); (2.2) is called a tightening of (2.1).

- for a minimization problems, optimal value of a relaxation yields a lower bound on the optimum
- relaxing integrality conditions of a MIP yields its LP relaxation

MIP

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax \geq b \\ & x_i \in \mathbb{Z} \quad \forall\, i \in N_1 \end{aligned}$$

LP relaxation

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax \geq b \end{aligned}$$

LP Relaxation of Node Cover Problem

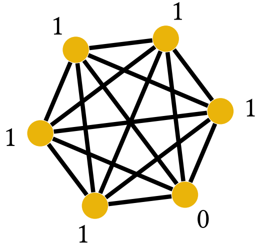
Node Cover IP

$$\begin{aligned} \min \quad & \sum_{v \in V} w_v\, x_v \\ \text{s.t.} \quad & x_v + x_{v'} \geq 1 \quad \forall \{v, v'\} \in E \\ & x_v \in \{0, 1\} \quad \forall\, v \in V \end{aligned}$$

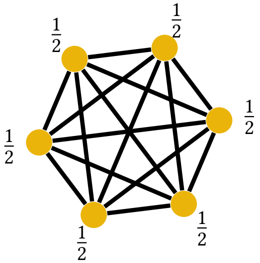
Node Cover LP relaxation

$$\begin{aligned} \min \quad & \sum_{v \in V} w_v\, x_v \\ \text{s.t.} \quad & x_v + x_{v'} \geq 1 \quad \forall \{v, v'\} \in E \\ & x_v \in [0, 1] \quad \forall\, v \in V \end{aligned}$$

Example: ‘integrality gap’ between IP and LP relaxation (for unit weights)



optimal IP solution of value 5



optimal LP solution of value 3

Introduction to

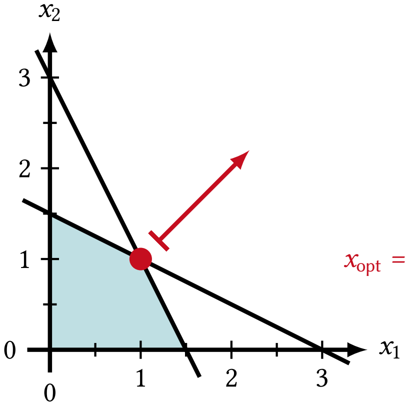
Linear and Combinatorial Optimization

2 Linear Programming Basics

2.3 Graphical Representation

2D example:

$$\begin{aligned} \min \quad & -x_1 - x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 3 \\ & 2x_1 + x_2 \leq 3 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

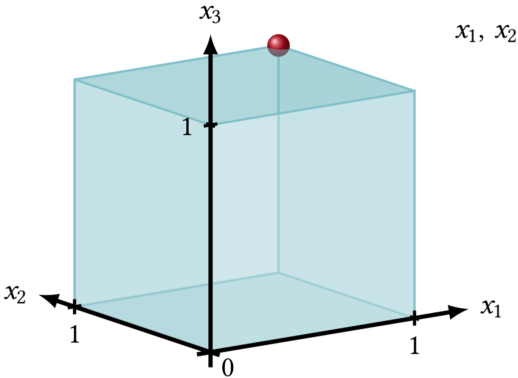


$x_{\text{opt}} = (1, 1)^T$

Graphical Representation and Solution (Cont.)

3D example:

$$\begin{array}{rrrrrrl} \min & -x_1 & - & x_2 & - & x_3 & \\ \text{s.t.} & x_1 & & & & & \leq 1 \\ & & & x_2 & & & \leq 1 \\ & & & & & x_3 & \leq 1 \\ & & & & x_1, & x_2, \ x_3 & \geq 0 \end{array}$$

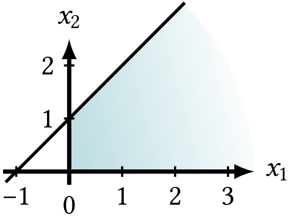


$x_{\text{opt}} = (1, 1, 1)^T$

Graphical Representation and Solution (Cont.)

another 2D example:

$$\begin{array}{lrcrl} \min & c_1 x_1 & + & c_2 x_2 & \\ \text{s.t.} & -x_1 & + & x_2 & \leq 1 \\ & & & x_1,\ x_2 & \geq 0 \end{array}$$



- for $c = (1,1)^\top$, the unique optimal solution is $x = (0,0)^\top$
- for $c = (1,0)^\top$, the optimal solutions are exactly the points

$$x = (0, x_2)^\top \qquad \text{with } 0 \leq x_2 \leq 1$$

- for $c = (0,1)^\top$, the optimal solutions are exactly the points

$$x = (x_1, 0)^\top \qquad \text{with } x_1 \geq 0$$

- for $c = (-1,-1)^\top$, the problem is unbounded, optimal cost is $-\infty$
- if we add the constraint $x_1 + x_2 \leq -1$, the problem is infeasible

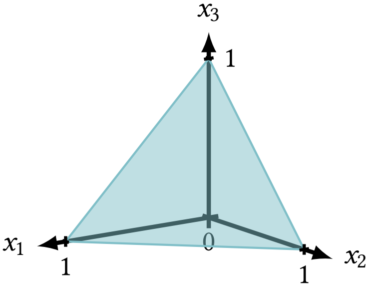
Properties of the Set of Optimal Solutions

In the last example, the following 5 cases occurred:

- i there is a unique optimal solution
- ii there exist infinitely many optimal solutions, but the set of optimal solutions is bounded
- iii there exist infinitely many optimal solutions and the set of optimal solutions is unbounded
- iv the problem is unbounded, i.e., the optimal cost is $-\infty$ and no feasible solution is optimal
- v the problem is infeasible, i.e., the set of feasible solutions is empty

These are indeed all cases that can occur in general (see also later).

Visualizing LPs in Standard Form



Example:

Let $A = (1, 1, 1) \in \mathbb{R}^{1\times 3}$, $b = (1) \in \mathbb{R}^1$ and consider the set of feasible solutions

$$P = \{x \in \mathbb{R}^3 \mid Ax = b,\ x \geq 0\}.$$

More general:

- if $A \in \mathbb{R}^{m\times n}$ with $m \leq n$ and the rows of A are linearly independent, then

$$\{x \in \mathbb{R}^n \mid A \cdot x = b\}$$

  is an $(n - m)$-dimensional affine subspace of $\mathbb{R}^n$.

- set of feasible solutions lies in this affine subspace and is only constrained by non-negativity constraints $x \geq 0$.

2 Linear Programming Basics

2.4 Piece-Wise Linear Objective

Piece-Wise Linear Objective

Linear Program

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax \geq b \end{aligned}$$

$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$,
$c \in \mathbb{R}^n$

Linear Program

with Piece-Wise Linear Objective

$$\begin{aligned} \text{minimize} \quad & \max_{i=1,\dots,k} \left\{ c_i^\top x + d_i \right\} \\ \text{subject to} \quad & Ax \geq b \end{aligned}$$

$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$,
$c_i \in \mathbb{R}^n$, $d_i \in \mathbb{R}$, $i = 1, \dots, k$

Example: Diet Problem with Diseconomies of Scale

- food 1 has cheap supplier able to procure $u_1 \in \mathbb{R}_{>0}$ units at price c_1 and expensive supplier able to procure ∞ units at price $C_1 > c_1$
- cost of purchasing x_1 units of food 1 becomes $\tilde{c}_1(x_1) = \max\left(c_1 x_1,\ C_1 x_1 - (C_1 - c_1)u_1\right)$
- Total costs of diet x:

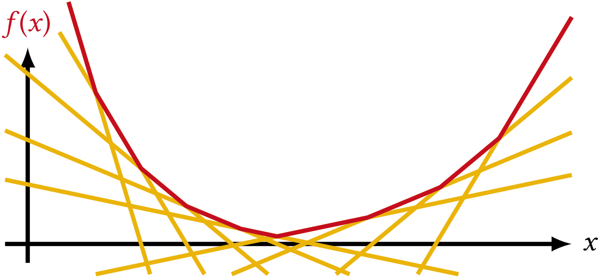
$$\tilde{c}(x) = \tilde{c}_1(x_1) + \sum_{i=2}^{n} c_i x_i = \max\left(c^\top x,\ c^\top x + (C_1 - c_1)(x_1 - u_1)\right)$$

Lemma 2.2

a An affine linear function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $f(x) = c^\top x + d$ with $c \in \mathbb{R}^n$, $d \in \mathbb{R}$, is both convex and concave.

b If $f_1, \dots, f_k : \mathbb{R}^n \longrightarrow \mathbb{R}$ are convex functions, then $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $f(x) := \max_{i=1,\dots,k} f_i(x)$ is also convex.

Example: The point-wise maximum of affine linear functions is convex.



Affine Linear Functions

Lemma 2.2

a An affine linear function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $f(x) = c^\top x + d$ with $c \in \mathbb{R}^n$, $d \in \mathbb{R}$, is both convex and concave.

b If $f_1, \ldots, f_k : \mathbb{R}^n \longrightarrow \mathbb{R}$ are convex functions, then $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $f(x) := \max_{i=1,\ldots,k} f_i(x)$ is also convex.

Proof: a For $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$:

$$\lambda \cdot f(x) + (1-\lambda) \cdot f(y) = \left(\lambda \cdot c^\top x + \lambda \cdot d\right) + \left((1-\lambda) \cdot c^\top y + (1-\lambda) \cdot d\right)$$

$$= c^\top\left(\lambda \cdot x + (1-\lambda) \cdot y\right) + \left(\lambda + (1-\lambda)\right) \cdot d = f\left(\lambda \cdot x + (1-\lambda) \cdot y\right)$$

b For $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$:

$$\lambda \cdot f(x) + (1-\lambda) \cdot f(y) = \lambda \cdot \max_{i=1,\ldots,k} f_i(x) + (1-\lambda) \cdot \max_{i=1,\ldots,k} f_i(y)$$

$$\geq \max_{i=1,\ldots,k} \left\{\lambda \cdot f_i(x) + (1-\lambda) \cdot f_i(y)\right\}$$

$$\geq \max_{i=1,\ldots,k} f_i\left(\lambda \cdot x + (1-\lambda) \cdot y\right) = f\left(\lambda \cdot x + (1-\lambda) \cdot y\right) \qquad \square$$

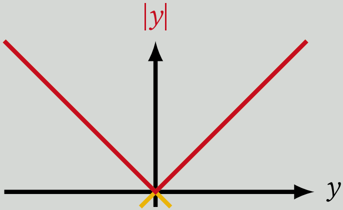
Piecewise Linear Convex Objective Functions

Let $c_1, \dots, c_k \in \mathbb{R}^n$ and $d_1, \dots, d_k \in \mathbb{R}$.

Consider piecewise linear convex function: $x \mapsto \max_{i=1,\dots,k} c_i^\top \cdot x + d_i$:

$$\begin{array}{ll} \min & \max\limits_{i=1,\dots,k} c_i^\top \cdot x + d_i \\ \text{s.t.} & A \cdot x \geq b \end{array} \quad \longleftrightarrow \quad \begin{array}{lll} \min & z & \\ \text{s.t.} & z \geq c_i^\top \cdot x + d_i & \text{for all } i \\ & A \cdot x \geq b & \end{array}$$

$$\begin{array}{ll} \min & \sum\limits_{i=1}^{n} c_i \cdot |x_i| \\ \text{s.t.} & A \cdot x \geq b \end{array} \quad \longleftrightarrow \quad \begin{array}{ll} \min & \sum\limits_{i=1}^{n} c_i \cdot z_i \\ \text{s.t.} & z_i \geq x_i \\ & z_i \geq -x_i \\ & A \cdot x \geq b \end{array}$$



Piecewise Linear Convex Objective Functions

Let $c_1, \ldots, c_k \in \mathbb{R}^n$ and $d_1, \ldots, d_k \in \mathbb{R}$.

Consider piecewise linear convex function: $x \mapsto \max_{i=1,\ldots,k} c_i^\top \cdot x + d_i$:

$$\begin{array}{ll} \min & \max\limits_{i=1,\ldots,k} c_i^\top \cdot x + d_i \\ \text{s.t.} & A \cdot x \geq b \end{array} \quad \longleftrightarrow \quad \begin{array}{lll} \min & z & \\ \text{s.t.} & z \geq c_i^\top \cdot x + d_i & \text{for all } i \\ & A \cdot x \geq b & \end{array}$$

$$\begin{array}{ll} \min & \sum\limits_{i=1}^{n} c_i \cdot |x_i| \\ \text{s.t.} & A \cdot x \geq b \end{array} \quad \longleftrightarrow \quad \begin{array}{ll} \min & \sum\limits_{i=1}^{n} c_i \cdot z_i \\ \text{s.t.} & z_i \geq x_i \\ & z_i \geq -x_i \\ & A \cdot x \geq b \end{array} \quad \longleftrightarrow \quad \begin{array}{ll} \min & \sum\limits_{i=1}^{n} c_i \cdot (x_i^+ + x_i^-) \\ \text{s.t.} & A \cdot (x^+ - x^-) \geq b \\ & x^+, x^- \geq 0 \end{array}$$