Introduction to

Linear and Combinatorial Optimization

3 Geometry of Linear Programming

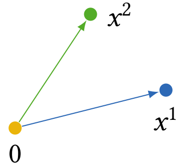
3.1 Hulls and Subspaces

Definition 3.1

Let $x^1, \dots, x^k \in \mathbb{R}^n$, $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$, and $x := \sum_{i=1}^k \lambda_i x^i$.

- a x is a linear combination of $x^1, \dots, x^k$.
- b If $\lambda \geq 0$, then x is a conic combination of $x^1, \dots, x^k$.
- c If $\sum_{i=1}^k \lambda_i = 1$, then x is an affine combination of $x^1, \dots, x^k$.
- d If $\lambda \geq 0$ and $\sum_{i=1}^k \lambda_i = 1$, x is a convex combination of $x^1, \dots, x^k$.

If $\lambda \notin \{0, e_1, \dots, e_n\}$, then x is a proper linear/conic/affine/convex combination.



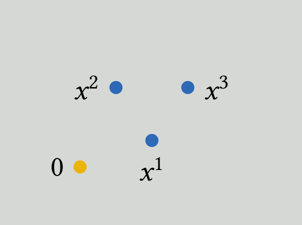
Definition 3.2 Let $\emptyset \neq S \subseteq \mathbb{R}^n$.

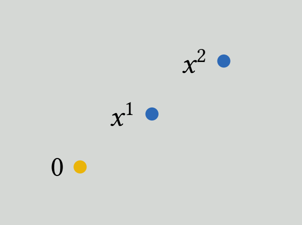
a the *linear hull* $\text{lin}(S)$ is the set of vectors that can be written as a linear combination of finitely many vectors from S; $\text{lin}(\emptyset) = \{0\}$.

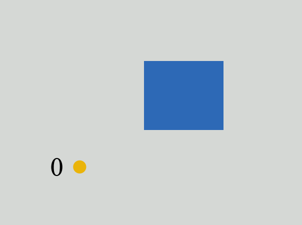
b the *conic hull* $\text{cone}(S)$ is the set of vectors that can be written as a conic combination of finitely many vectors from S; $\text{cone}(\emptyset) = \{0\}$

c the *affine hull* $\text{aff}(S)$ is the set of vectors that can be written as an affine combination of finitely many vectors from S; $\text{aff}(\emptyset) = \emptyset$

d the *convex hull* $\text{conv}(S)$ is the set of vectors that can be written as a convex combination of finitely many vectors from S; $\text{conv}(\emptyset) = \emptyset$







Definition 3.3

a. $S \subseteq \mathbb{R}^n$ is a linear subspace of $\mathbb{R}^n$, if $S = \text{lin}(S)$.
b. $S \subseteq \mathbb{R}^n$ is a (convex) cone, if $S = \text{cone}(S)$.
c. $S \subseteq \mathbb{R}^n$ is an affine subspace of $\mathbb{R}^n$, if $S = \text{aff}(S)$.
d. $S \subseteq \mathbb{R}^n$ is called convex, if $S = \text{conv}(S)$.

Remark

- in Definition 1.1, we defined a set $S \subseteq \mathbb{R}^n$ to be convex if $\lambda x + (1-\lambda)y \in S$ for all $x, y \in S, \lambda \in [0, 1]$
- both definitions are equivalent:
  - S convex wrt. Def. 3.3 $\Longrightarrow$ S convex wrt. Def. 1.1 is obvious
  - let S convex wrt. Def. 1.1, $x^1, \dots, x^k \in S$ and $x = \sum_{i=1}^k \lambda_i x^i$, $\sum_{i=1}^k \lambda_i = 1, \lambda \geq 0$.
  - We show by induction on k that $x \in S$; (This is trivial for $k = 1$).
  - So assume all convex combinations up to $k-1$ elements of S lie in S. Then, $x = \sum_{i=1}^k \lambda_i x^i = (1-\lambda_k)y + \lambda_k x^k$, with $y := \sum_{i=1}^{k-1} \frac{\lambda_i}{1-\lambda_k} x^i$
  - y is a convex combination of $k-1$ elements, so $y \in S$ by induction.
  - Hence $x \in S$ by Def. 1.1

Lemma 3.4 Let $S \subseteq \mathbb{R}^n$.

$$\left\{\begin{matrix}\text{lin}(S)\\ \text{cone}(S)\\ \text{aff}(S)\\ \text{conv}(S)\end{matrix}\right\} \text{ is the } \subseteq\text{-smallest } \left\{\begin{matrix}\text{linear subspace}\\ \text{cone}\\ \text{affine subspace}\\ \text{convex set}\end{matrix}\right\} \text{ containing } S.$$

Proof:

- $\text{conv}(S)$ is a convex set since $\text{conv}\big(\text{conv}(S)\big) = \text{conv}(S)$
- for any convex set $X \supseteq S$, it holds that $X = \text{conv}(X) \supseteq \text{conv}(S)$
- similar for $\text{lin}(S)$, $\text{cone}(S)$, and $\text{aff}(S)$ □

Lemma 3.5 Families of linear subspaces / cones / affine subspaces / convex sets in $\mathbb{R}^n$ are closed under taking intersections.

Proof: Let $(C_j)_{j\in\mathcal{J}}$ be a family of cones, $\mathcal{J}$ arbitrary. We show $\text{cone}\big(\bigcap_{j\in\mathcal{J}} C_j\big) = \bigcap_{j\in\mathcal{J}} C_j$.

- "$\supseteq$" is obvious
- for "$\subseteq$": $\text{cone}\big(\bigcap_{j\in\mathcal{J}} C_j\big) \subseteq \bigcap_{j\in\mathcal{J}} \text{cone}\big(C_j\big) = \bigcap_{j\in\mathcal{J}} C_j$
- similar for lin, aff, and conv □

Definition 3.6 A finite non-empty subset $S \subseteq \mathbb{R}^n$ is linearly (affinely) independent, if no element of S can be written as a proper linear (affine) combination of elements from S.

Definition 3.7 The dimension $\dim(S)$ of a subset $S \subseteq \mathbb{R}^n$ is the largest cardinality of an affinely independent subset of S minus 1.

Remarks

- for a linear subspace $S \subseteq \mathbb{R}^n$, $\dim(S)$ according to Definition 3.7 is equal to the maximal number of linearly independent vectors in S
  - "≥": adding the zero-vector to a linearly independent set of vectors yields an affinely independent set
  - "≤": if $v_1, \ldots, v_k, v_{k+1}$ are affinely independent, then $v_1 - v_{k+1}, \ldots, v_k - v_{k+1}$ are linearly independent

Introduction to

Linear and Combinatorial Optimization

3 Geometry of Linear Programming

3.2 Hyperplanes and Halfspaces

Hyperplanes and Halfspaces

Definition 3.8 Let $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$.

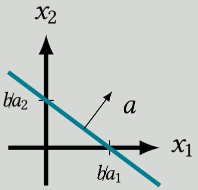
a $\{x \in \mathbb{R}^n \mid a^\top \cdot x = b\}$ is called hyperplane

b $\{x \in \mathbb{R}^n \mid a^\top \cdot x \geq b\}$ is called halfspace

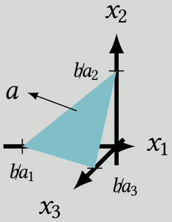
Remarks

- A hyperplane is an affine subspace of dimension $n - 1$.
- A hyperplane with right-hand side $b = 0$ is a linear subspace.
- Hyperplanes and halfspaces are cones if and only if $b = 0$.
- Hyperplanes and halfspaces are convex sets.

in dimension 2

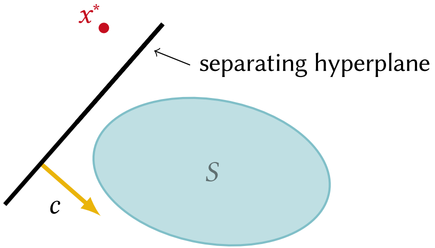


in dimension 3



Separating Hyperplane Theorem for Convex Sets

Theorem 3.9 Let $S \subseteq \mathbb{R}^n$ closed and convex, and let $x^* \in \mathbb{R}^n \setminus S$. There exists a vector $c \in \mathbb{R}^n$ such that $c^\top \cdot x^* < c^\top \cdot x$ for all $x \in S$.



Corollary 3.10 Every closed and convex set is the intersection of a family of halfspaces.

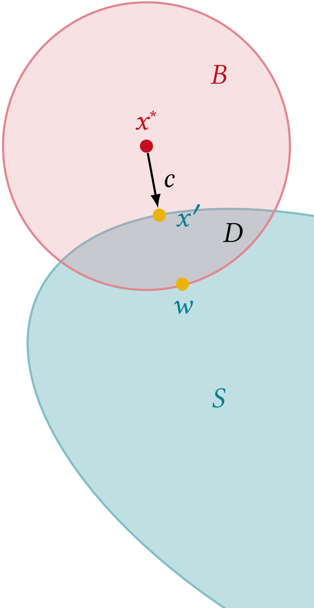
Proof of Theorem 3.9

For arbitrary $w \in S$ let $B := \left\{ x \,\middle|\, \|x - x^*\| \leq \|w - x^*\| \right\}, \quad \|.\| := \|.\|_2$

$\implies \quad D := S \cap B$ closed, bounded, non-empty

$\implies \quad \exists\, x' \in D : \; \|x' - x^*\| \leq \|x - x^*\| \quad \forall\, x \in D$

$\implies \quad \|x' - x^*\| \leq \|x - x^*\| \quad \forall\, x \in S \qquad (*)$



Proof of Theorem 3.9

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$\Longrightarrow \quad D := S \cap B$ closed, bounded, non-empty

$\Longrightarrow \quad \exists\, x' \in D : \; \|x' - x^*\| \leq \|x - x^*\| \quad \forall\, x \in D$

$\Longrightarrow \quad \|x' - x^*\| \leq \|x - x^*\| \quad \forall\, x \in S \qquad (*)$

Claim: $c^\top x > c^\top x^*$ with $c := x' - x^* \neq 0 \; \forall\, x \in S$

Proof:

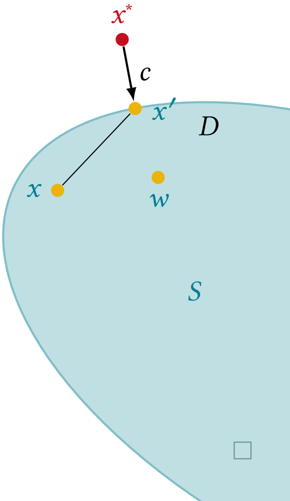
For all $x \in S$ and $\lambda \in (0, 1]$: $\quad x' + \lambda(x - x') \in S$

$$\Longrightarrow \quad \|x' - x^*\|^2 \overset{(*)}{\leq} \|x' + \lambda(x - x') - x^*\|^2$$
$$= \|x' - x^*\|^2 + 2\lambda(x' - x^*)^\top(x - x') + \lambda^2\|x - x'\|^2$$

$$\Longrightarrow \quad 0 \leq (x' - x^*)^\top(x - x') + \tfrac{1}{2}\lambda\|x - x'\|^2$$

$$\overset{\lambda \to 0}{\Longrightarrow} \quad 0 \leq (x' - x^*)^\top(x - x') = c^\top(x - x')$$

$$\Longrightarrow \quad c^\top x \geq c^\top x' = c^\top x^* + c^\top(x' - x^*) = c^\top x^* + \|c\|^2 > c^\top x^*$$



□

Introduction to

Linear and Combinatorial Optimization

3 Geometry of Linear Programming

3.3 Polyhedra and Polytopes

Definition 3.11 Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

a $\left\{x \in \mathbb{R}^n \mid A \cdot x \geq b\right\}$ is called polyhedron

b $\left\{x \in \mathbb{R}^n \mid A \cdot x = b,\ x \geq 0\right\}$ is a polyhedron in standard form representation

- a polyhedron is an intersection of finitely many halfspaces.

Definition 3.12

a Set $S \subseteq \mathbb{R}^n$ is bounded if there is $K \in \mathbb{R}$ such that

$$\|x\|_\infty \leq K \qquad \text{for all } x \in S.$$

b A bounded polyhedron is called polytope.

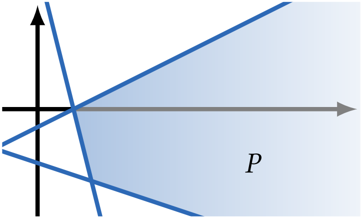
Polyhedral Cones and Recession Cones

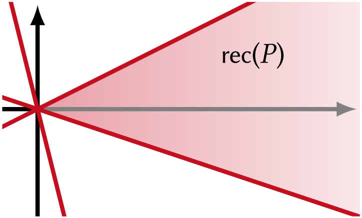
Definition 3.13 For $A \in \mathbb{R}^{m\times n}$ the polyhedron $\left\{ x \in \mathbb{R}^n \mid A \cdot x \geq 0 \right\}$ is a polyhedral cone.

- cone $\{0\}$ is the only polyhedral cone in $\mathbb{R}^n$ that is a polytope.

Definition 3.14 For a non-empty polyhedron $P = \left\{ x \in \mathbb{R}^n \mid A \cdot x \geq b \right\}$, the recession cone $\mathrm{rec}(P)$ is the polyhedral cone $\left\{ x \in \mathbb{R}^n \mid A \cdot x \geq 0 \right\}$. The non-zero elements of the recession cone are the rays of P.

Example in $\mathbb{R}^2$:





Characterization of Recession Cones

Lemma 3.15 For a non-empty polyhedron $P = \{x \in \mathbb{R}^n \mid A \cdot x \geq b\}$ and $y \in P$,

$$\operatorname{rec}(P) = \left\{ x \in \mathbb{R}^n \mid y + \lambda \cdot x \in P \text{ for all } \lambda \geq 0 \right\}.$$

In particular, if P is a polytope, $\operatorname{rec}(P) = \{0\}$.

Proof: For a non-empty polyhedron $P = \left\{ x \in \mathbb{R}^n \mid A \cdot x \geq b \right\}$ and $y \in P$,

$$\left\{ x \in \mathbb{R}^n \mid y + \lambda \cdot x \in P \; \forall \lambda \geq 0 \right\} = \left\{ x \in \mathbb{R}^n \mid A \cdot (y + \lambda \cdot x) \geq b \; \forall \lambda \geq 0 \right\}$$
$$= \left\{ x \in \mathbb{R}^n \mid A \cdot x \geq 0 \right\}.$$

□

- the recession cone of $P = \left\{ x \in \mathbb{R}^n \mid A \cdot x = b,\ x \geq 0 \right\}$ is

$$\left\{ x \in \mathbb{R}^n \mid A \cdot x = 0,\ x \geq 0 \right\}$$

- proof via $\{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\} = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{bmatrix} A \\ -A \\ I \end{bmatrix} x \geq \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix} \right\}$

Active and Binding Constraints

In the following, let $P \subseteq \mathbb{R}^n$ be a polyhedron defined by

$$a_i^\top \cdot x \geq b_i \qquad \text{for } i \in M,$$
$$a_i^\top \cdot x = b_i \qquad \text{for } i \in N,$$

with $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$, for all i.

Definition 3.16 If $x^* \in \mathbb{R}^n$ satisfies $a_i^\top \cdot x^* = b_i$ for some i, then the corresponding constraint is active (or binding) at x^*.

Lemma 3.17 (Dimension Lemma)
If $P \neq \emptyset$ and none of the constraints $a_i^\top \cdot x \geq b_i$ for $i \in M$ is active for all $x \in P$, then $\dim(P) = n - \operatorname{rank}\left\{ a_i \mid i \in N \right\}$.

Proof:

- let A_N be the matrix consisting of the rows $a_i^\top$, $i \in N$
- rank-nullity theorem: $n = \operatorname{rank}(A_N) + \dim(\ker(A_N))$
- left to show: $\dim(P) = \dim(\ker(A_N))$

Proof of Lemma 3.17 (Cont.)

$\dim(\ker(A_N)) \geq \dim(P) =: k$:

- there are $k+1$ affinely independent vectors $x_0, x_1, \dots, x_k \in P$
- $x_1 - x_0, \dots, x_k - x_0$ are linearly independent and satisfy $A_N(x_i - x_0) = 0$ for all $i = 1, \dots, k$
- $\dim(\ker(A_N)) \geq \dim(P)$

$l := \dim(\ker(A_N)) \leq \dim(P)$:

- let $x_j \in \mathbb{R}^n$, $j = 1, \dots, l$, linearly independent with $A_N x_j = 0 \, \forall j$
- for $i \in M$, let $y^i \in P$ with $a_i^\top \cdot y^i > b_i$
- let $y := \frac{1}{|M|} \sum_{i \in M} y^i \in P$, then $a_i^\top \cdot y > b_i$ for all $i \in M$
- for $\varepsilon > 0$ small enough, $a_i^\top \cdot (y + \varepsilon\, x_j) \geq b_i$ for all $i \in M$, $j = 1, \dots, l$.
- $y + \varepsilon\, x_j \in P$ for $j = 1, \dots, l$
- $\{y\} \cup \left\{ y + \varepsilon\, x_j \mid j \in \{1, \dots, l\} \right\}$ affinely independent
- $\dim(P) \geq l$. □

Introduction to

Linear and Combinatorial Optimization

3 Geometry of Linear Programming

3.4 Vertices and Extreme Points

Definition 3.18 Let $P \subseteq \mathbb{R}^n$ be a polyhedron.

a $x \in P$ is an extreme point of P if

$$x \neq \lambda \cdot y + (1 - \lambda) \cdot z \qquad \text{for all } y, z \in P \setminus \{x\}, 0 \leq \lambda \leq 1,$$

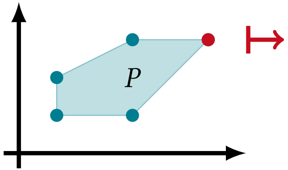
i.e., x is not a convex combination of two other points in P.

b $x \in P$ is a vertex of P if there is some $c \in \mathbb{R}^n$ such that

$$c^\top \cdot x < c^\top \cdot y \qquad \text{for all } y \in P \setminus \{x\},$$

i.e., x is the unique optimal solution to the LP $\min\{c^\top \cdot z \mid z \in P\}$.

- the only possible extreme point or vertex of a polyhedral cone C is $0 \in C$



Theorem 3.19 Let $x^* \in \mathbb{R}^n$ and $I = \{i \mid a_i^\top \cdot x^* = b_i\}$. The following are equivalent:

- i There are n vectors in $\{a_i \mid i \in I\}$ which are linearly independent.
- ii The vectors in $\{a_i \mid i \in I\}$ span $\mathbb{R}^n$, i.e. $\text{lin}(\{a_i \mid i \in I\}) = \mathbb{R}^n$.
- iii x^* is the unique solution to the system of equations $a_i^\top \cdot x = b_i$, $i \in I$.

Characterization of Vertices and Extreme Points

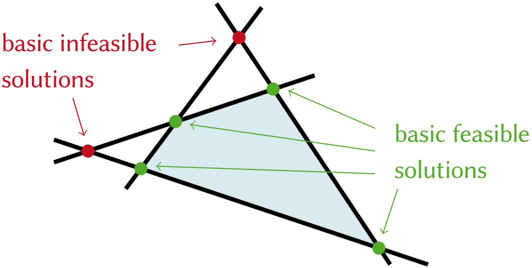
Definition 3.20

a $x^* \in \mathbb{R}^n$ is a basic solution of P if

- all equality constraints are active and
- there are n linearly independent constraints that are active.

b A basic solution satisfying all constraints is a basic feasible solution.

Example:



Characterization of Vertices and Extreme Points

Definition 3.20

a $x^* \in \mathbb{R}^n$ is a basic solution of P if

- all equality constraints are active and
- there are n linearly independent constraints that are active.

b A basic solution satisfying all constraints is a basic feasible solution.

Theorem 3.21 For $x^* \in P$, the following are equivalent:

i x^* is a vertex of P.

ii x^* is an extreme point of P.

iii x^* is a basic feasible solution of P.

We assume that $P \subseteq \mathbb{R}^n$ is a polyhedron defined by

$$a_i^\top \cdot x \geq b_i \qquad \text{with } a_i \in \mathbb{R}^n, b_i \in \mathbb{R} \text{ for } i \in M,$$
$$a_i^\top \cdot x = b_i \qquad \text{with } a_i \in \mathbb{R}^n, b_i \in \mathbb{R} \text{ for } i \in N.$$

Proof of Theorem 3.21: (iii) $\implies$ (i)

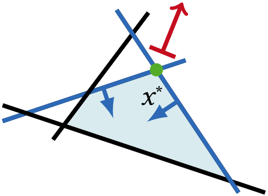
- x^* basic feasible solution, $I := \left\{ i \in M \cup N \mid a_i^\top \cdot x^* = b_i \right\}$
- for $c := \sum_{i \in I} a_i$, we have

$$c^\top \cdot x^* = \sum_{i \in I} a_i^\top \cdot x^* = \sum_{i \in I} b_i$$

$$c^\top \cdot x = \sum_{i \in I} a_i^\top \cdot x \geq \sum_{i \in I} b_i \quad \text{for all } x \in P$$

- since there are n linearly independent vectors in $\{a_i \mid i \in I\}$:

$$x \in P, c^\top \cdot x = \sum_{i \in I} b_i \iff a_i^\top \cdot x = b_i \; \forall\, i \in I \iff x = x^*$$



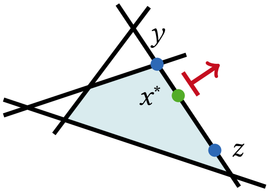
Proof of Theorem 3.21: (i) $\Longrightarrow$ (ii)

- x^* vertex $\Longrightarrow \exists\, c \in \mathbb{R}^n : \; c^\top \cdot x^* < c^\top \cdot y \quad \forall\, y \in P \setminus \{x^*\}$
- for a contradiction assume x^* is *not* an extreme point, i.e.,

$$x^* = \lambda y + (1-\lambda) z \qquad \text{with } y, z \in P \setminus \{x^*\},\; \lambda \in [0,1].$$

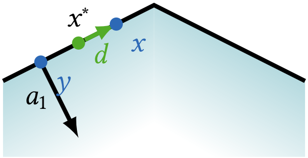
- then,

$$c^\top \cdot x^* = \lambda \underbrace{c^\top \cdot y}_{> c^\top \cdot x^*} + (1-\lambda) \underbrace{c^\top \cdot z}_{> c^\top \cdot x^*} > c^\top \cdot x^* \quad ⚡$$



Proof of Theorem 3.21: (ii) $\Longrightarrow$ (iii)

- x^* extreme point, $I := \left\{ i \in M \cup N \mid a_i^\top \cdot x^* = b_i \right\}$
- assume by contradiction that $\operatorname{rank}\{a_i \mid i \in I\} < n$
- there exists $d \in \mathbb{R}^n \setminus \{0\}$ with $a_i^\top \cdot d = 0$ for all $i \in I$
- let $x := x^* + \varepsilon d$ and $y := x^* - \varepsilon d$ for some $\varepsilon > 0$
- we claim that $x, y \in P$ for $\varepsilon > 0$ small enough
  - for $i \in I$, $a_i^\top \cdot x = \underbrace{a_i^\top \cdot x^*}_{=b_i} + \varepsilon \underbrace{a_i^\top \cdot d}_{=0} = b_i$
  - for $i \notin I$, $a_i^\top \cdot x = \underbrace{a_i^\top \cdot x^*}_{>b_i} + \varepsilon\, a_i^\top \cdot d \geq b_i$ for $\varepsilon > 0$ small enough
  - the same holds for y instead of x
- $x^* = \frac{x+y}{2}$ ↯ $\square$



Existence of Extreme Points

Definition 3.22 A polyhedron P is called pointed if it contains at least one vertex.

Definition 3.23 A polyhedron $P \subseteq \mathbb{R}^n$ contains a line if there is $x \in P$ and a direction $d \in \mathbb{R}^n \setminus \{0\}$ such that

$$x + \lambda \cdot d \in P \qquad \text{for all } \lambda \in \mathbb{R}.$$

Theorem 3.24 Consider non-empty $P \subseteq \mathbb{R}^n$. The following are equivalent:

i P is pointed.

ii P does not contain a line.

iii $\operatorname{rank}\{a_i \mid i \in M \cup N\} = n$.

Proof of Theorem 3.24

Thm 3.24: i P pointed $\iff$ ii P contains no line $\iff$ iii $\operatorname{rank}(A_{M\cup N}) = n$

i $\implies$ iii: $x^*\in P$ vertex $\implies$ x^* basic feasible solution

$\implies$ There are n linearly independent constraints that are active at x^*.

$\implies$ There are n linearly independent vectors in $\{a_i \mid i\in M\cup N\}$.

iii $\implies$ ii: By contradiction assume that P contains a line.

That is, there are $x\in P$, $d\in\mathbb{R}^n\setminus\{0\}$ with

$$x+\lambda d\in P \qquad \text{for all } \lambda\in\mathbb{R}.$$

$\implies \quad a_i^\top\cdot(x+\lambda d) = a_i^\top\cdot x+\lambda a_i^\top\cdot d \geq b_i \;\; \forall i\in M\cup N,\ \lambda\in\mathbb{R}$

$\implies \quad a_i^\top\cdot d = 0 \;\; \forall i\in M\cup N.$

$\implies \quad \operatorname{rank}\{a_i \mid i\in M\cup N\} < n.$ ↯

Proof of Theorem 3.24 (Cont.)

Thm 3.24: i P pointed $\iff$ ii P contains no line $\iff$ iii $\operatorname{rank}(A_{M \cup N}) = n$

ii $\Longrightarrow$ i: Choose $x \in P$ maximizing $|I|$ with $N \subseteq I := \left\{ i \in M \cup N \mid a_i^\top \cdot x = b_i \right\}$.

If $\operatorname{rank}\{ a_i \mid i \in I \} = n$, then x is a vertex and we are done.

Otherwise, there is $d \in \mathbb{R}^n \setminus \{0\}$ with $a_i^\top \cdot d = 0$ for all $i \in I$.

$\Longrightarrow \quad a_i^\top \cdot (x + \lambda\, d) = a_i^\top \cdot x = b_i \quad$ for all $i \in I, \lambda \in \mathbb{R}$.

Since P does not contain a line, $x + \lambda'\, d \notin P$ for some $\lambda' \in \mathbb{R}$.

$\Longrightarrow \quad a_i^\top \cdot (x + \lambda'\, d) < b_i$ for some $i \in M \setminus I$.

Assume w.l.o.g. $\lambda' > 0$ (otherwise replace d with $-d$) and let

$$\lambda_0 := \max\left\{ \lambda \mid a_i^\top \cdot (x + \lambda\, d) \geq b_i \ \forall\, i \in M \cup N \right\} < \lambda'.$$

But then, there is an $i \in M \setminus I$ with $a_i^\top \cdot (x + \lambda_0\, d) = b_i$.

$\Longrightarrow$ At least $|I| + 1$ constraints active at $x + \lambda_0\, d \in P$. ⚡ □

Existence of Extreme Points (Cont.)

Corollary 3.25

a. A non-empty polytope contains an extreme point.

b. A non-empty polyhedron in standard form contains an extreme point.

c. Polyhedron $P \neq \emptyset$ is pointed if and only if $\text{rec}(P)$ is pointed.

Proof: of b: P cannot contain a line because of the constraint $x \geq 0$:

$$x + \lambda d \geq 0, \forall \lambda \in \mathbb{R} \implies d = 0.$$

Proof: of c: P has a line of direction $d \neq 0$

$$\begin{aligned} &\iff d \in \text{rec}(P), -d \in \text{rec}(P) \\ &\iff \{\lambda d : \lambda \in \mathbb{R}\} \text{ is a line of } \text{rec}(P) \\ &\iff \text{rec}(P) \text{ has a line of dir. } d \neq 0 \end{aligned}$$

□

Example:

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{rcrl} x_1 & + & x_2 & \geq 1 \\ x_1 & + & 2\,x_2 & \geq 0 \end{array} \right\}$$

contains a line since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in P$ for all $\lambda \in \mathbb{R}$.

Characterization of Polytopes

Theorem 3.26 A polytope is equal to the convex hull of its vertices.

Proof: Follows from the equivalence of vertices, extreme points and basic feasible solutions (Theorem 3.21); see exercise. □

Theorem 3.27 A set $P \subseteq \mathbb{R}^n$ is a polytope if and only if there exists a finite set $V \subseteq \mathbb{R}^n$ such that P is the convex hull of V.

Proof: '$\Longrightarrow$': Follows from Theorem 3.26 since a polytope has only finitely many basic feasible solutions (vertices).

'$\Longleftarrow$': See exercise session. □

Introduction to

Linear and Combinatorial Optimization

3 Geometry of Linear Programming

3.5 Faces and Facets

Faces and Facets

Definition 3.28 Let $P \subseteq \mathbb{R}^n$ be a polyhedron, $c \in \mathbb{R}^n \setminus \{0\}$ and $\gamma \in \mathbb{R}$.

a The linear inequality $c^\top \cdot x \geq \gamma$ is valid for P if $P \subseteq \left\{ x \mid c^\top \cdot x \geq \gamma \right\}$.

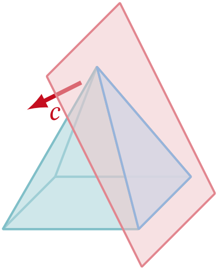
b The hyperplane $H = \left\{ x \in \mathbb{R}^n \mid c^\top \cdot x = \gamma \right\}$ is a supporting hyperplane of P if $c^\top \cdot x \geq \gamma$ is valid for P and $P \cap H \neq \emptyset$.

c The intersection of P with a supporting hyperplane is a face of P. Also P and $\emptyset$ are faces of P; the others are called proper faces.

d The inclusion-wise maximal proper faces are called facets.

Remarks

- Every face of P is itself a polyhedron.
- Every vertex of P is a 0-dimensional face of P.
- The optimal LP solutions form a face of the underlying polyhedron.



Characterization of Faces

Theorem 3.29 Consider a polyhedron $P \subseteq \mathbb{R}^n$ be defined by

$$a_i^\top \cdot x \geq b_i \qquad \text{for } i \in M,$$
$$a_i^\top \cdot x = b_i \qquad \text{for } i \in N,$$

and let $F \neq \emptyset$ be a face of P.

a There exists $K \subseteq M$ with $F = \left\{ x \in P \mid a_i^\top \cdot x = b_i \text{ for all } i \in K \right\}$.

b For $K \subseteq M$, the subset $\left\{ x \in P \mid a_i^\top \cdot x = b_i \text{ for all } i \in K \right\}$ is a face of P.

c $G \subseteq F$ is a face of F if and only if it is a face of P.

d There is a chain of faces $F = F_0 \subset F_1 \subset \cdots \subset F_q = P$ such that $\dim(F_{i+1}) = \dim(F_i) + 1$, for $i = 0, \ldots, q-1$.

Proof of Theorem 3.29 (a)

a $\exists K \subseteq M : F = \{x \in P \mid a_i^\top x = b_i \; \forall i \in K\}$

Let $K := \left\{ i \in M \mid a_i^\top \cdot x = b_i \text{ for all } x \in F \right\}$.

Claim: $F = \left\{ x \in P \mid a_i^\top \cdot x = b_i \text{ for all } i \in K \right\}$

'$\subseteq$': Clear by definition of K.

'$\supseteq$': Assume by contradiction that $y \in P \setminus F$ with $a_i^\top \cdot y = b_i \; \forall \, i \in K$.

Let $c^\top \cdot x \geq \gamma$ valid for P such that $F = \left\{ x \in P \mid c^\top \cdot x = \gamma \right\}$.

In particular, $c^\top \cdot y > \gamma$ as $y \in P \setminus F$.

For each $i \in M \setminus K$ there is an $x^i \in F$ with $a_i^\top \cdot x^i > b_i$.

Let $x_0 := \frac{1}{|M \setminus K|} \sum_{i \in M \setminus K} x^i \in F$ (convex), thus $c^\top \cdot x_0 = \gamma$.

Notice that $a_i^\top \cdot x_0 = b_i \; \forall \, i \in K$ and $a_i^\top \cdot x_0 > b_i \; \forall \, i \in M \setminus K$.

For $\varepsilon > 0$ small enough, $z := x_0 + \varepsilon(x_0 - y) \in P$ because:

$$a_i^\top \cdot z = (1 + \varepsilon)\, a_i^\top \cdot x_0 - \varepsilon\, a_i^\top \cdot y \begin{cases} = b_i & \text{for } i \in N \cup K, \\ \geq b_i & \text{for } i \in M \setminus K \end{cases}$$

But $c^\top \cdot z = (1 + \varepsilon)\, c^\top \cdot x_0 - \varepsilon\, c^\top \cdot y < \gamma$. ↯

Proof of Theorem 3.29 (b)–(d)

b $\{x \in P \mid a_i^\top x = b_i \ \forall i \in K\}$ is a face $\forall K \subseteq M$

Let $c := \sum_{i \in K} a_i$ and $\gamma := \sum_{i \in K} b_i$.

Then, $c^\top \cdot x \geq \gamma$ is a valid inequality for P and for $x \in P$

$$c^\top \cdot x = \gamma \qquad \Longleftrightarrow \qquad a_i{}^\top \cdot x = b_i \text{ for all } i \in K.$$

c G face of $F \Leftrightarrow G$ face of $P \quad \forall G \subseteq F$

'$\Longleftarrow$': If $G = \left\{x \in P \mid c^\top \cdot x = \gamma\right\} \subseteq F$ with $c^\top \cdot x \geq \gamma$ valid for P,
then $G = \left\{x \in F \mid c^\top \cdot x = \gamma\right\}$ and $c^\top \cdot x \geq \gamma$ valid for F.

'$\Longrightarrow$': $F = \left\{x \mid a_i^\top \cdot x \geq b_i \ \forall\, i \in M \setminus K,\ a_i^\top \cdot x = b_i \ \forall\, i \in K \cup N\right\}$
for some $K \subseteq M$ due to a. Since G is a face of F, again due to a,
$G = \left\{x \mid a_i^\top \cdot x \geq b_i \ \forall\, i \in M \setminus L,\ a_i^\top \cdot x = b_i \ \forall\, i \in L \cup N\right\}$
for some $K \subseteq L \subseteq M$. Thus, due to b, G is a face of P.

d $F_0 \subset F_1 \subset \ldots F_q = P$ with $\dim(F_{i+1}) = \dim(F_i)+1$, $i=0,\ldots,q-1$

Follows from a–c and the Dimension Lemma (Lemma 3.17) □

Characterization of Faces (Cont.)

Corollary 3.30 Consider a polyhedron $P \subseteq \mathbb{R}^n$ be defined by

$$\begin{aligned} a_i^\top \cdot x &\geq b_i && \text{for } i \in M, \\ a_i^\top \cdot x &= b_i && \text{for } i \in N. \end{aligned}$$

- a P has finitely many distinct faces.
- b If F is a facet of P, then $\dim(F) = \dim(P) - 1$.
- c An inclusion-wise minimal proper face F of P can be written as

$$F = \left\{ x \in \mathbb{R}^n \mid a_i^\top \cdot x = b_i \text{ for all } i \in K \cup N \right\}$$

  for some $K \subseteq M$ with $\operatorname{rank}\{a_i \mid i \in K \cup N\} = \operatorname{rank}\{a_i \mid i \in M \cup N\}$.
- d If P is pointed, every minimal nonempty face of P is a vertex.

Proof: Exercise. □

Number of Vertices

Corollary 3.31

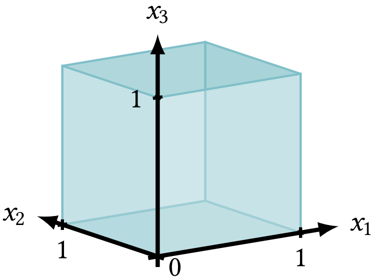
a. A polyhedron has a finite number of vertices and basic solutions.

b. For a polyhedron in $\mathbb{R}^n$ given by m linear inequalities, this number is at most $\binom{m}{n}$.

Example:

$P := \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1,\ i = 1, \ldots, n\}$ (n-dimensional unit cube)

- number of constraints: $m = 2n$
- number of vertices: 2^n
  note that $\binom{2n}{n} = \frac{(n+1)}{1}\frac{n+2}{2} \cdots \frac{2n}{n} \geq 2^n$



Theorem 3.32 Let $P \subseteq \mathbb{R}^n$ a pointed polyhedron and $c \in \mathbb{R}^n$.
If $\min\left\{ c^\top \cdot x \mid x \in P \right\}$ is bounded, there is a vertex that is optimal.

Corollary 3.33 Every linear programming problem is either infeasible or unbounded or there exists an optimal solution.

Proof:

- every linear program is equivalent to an LP in standard form
- every polyhedron in standard form is pointed (Corollary 3.25)
- Theorem 3.32 implies the result □

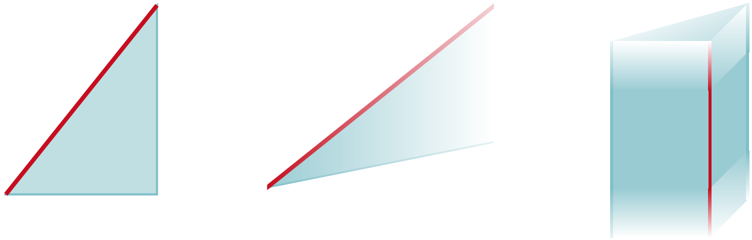
Edges, Extreme Rays, Extreme Lines

Definition 3.34 A one-dimensional face F of polyhedron P is

a. an edge if F has two vertices,
   i.e., $F = \operatorname{conv}(\{x, y\})$ with $x, y \in \mathbb{R}^n$, $x \neq y$;

b. an extreme ray if F has one vertex,
   i.e., $F = x + \operatorname{cone}(\{z\})$ with $x \in \mathbb{R}^n$, $z \in \mathbb{R}^n \setminus \{0\}$;

c. an extreme line if F has no vertex,
   i.e., $F = x + \operatorname{lin}(\{z\})$ with $x \in \mathbb{R}^n$, $z \in \mathbb{R}^n \setminus \{0\}$.



Adjacent Basic Solutions and Edges

Definition 3.35 Let $P \subseteq \mathbb{R}^n$ be a polyhedron. Two distinct basic solutions are adjacent if there are $n - 1$ linearly independent constraints that are active at both of them.

Observation 3.36 Let $x, y \in P$ with $x \neq y$ be two adjacent basic feasible solutions, then the line segment $\text{conv}(\{x, y\})$ that joins them is an edge of P.

Proof:

- for $z \in \{x, y\}$, let $I(z) := \{i \in M \cup N \mid a_i^\top z = b_i\}$
- let $I := I(x) \cap I(y)$, then $|I| = n - 1$
- by dimension Lemma 3.17, $F = \{x \in P \mid a_i^\top x = b_i \; \forall i \in I\}$ is a face of dimension 1
- again by Theorem 3.29, x and y are vertices of F, thus, by Theorem 3.26, $F = \text{conv}(\{x, y\})$ □

Introduction to

Linear and Combinatorial Optimization

3 Geometry of Linear Programming

3.6 Polyhedra in Standard Form

Polyhedra in Standard Form

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $P = \{x \in \mathbb{R}^n \mid A \cdot x = b,\ x \geq 0\}$ a polyhedron in standard form representation.

Observation.

One can assume without loss of generality that $\text{rank}(A) = m$.

Proof: Let $a_1^\top, \ldots, a_m^\top \in \mathbb{R}^n$ rows of A. Assume that $a_i = \sum_{j \neq i} \lambda_j \cdot a_j$.

Case 1: $b_i = \sum_{j \neq i} \lambda_j\, b_j$. Then,

$$a_j^\top \cdot x = b_j \quad \forall\, j \neq i \quad \Longrightarrow \quad a_i^\top \cdot x = \sum_{j \neq i} \lambda_j \cdot (a_j^\top \cdot x) = \sum_{j \neq i} \lambda_j\, b_j = b_i.$$

Thus the ith constraint is redundant and can be deleted.

Case 2: $b_i \neq \sum_{j \neq i} \lambda_j\, b_j \quad \Longrightarrow \quad A \cdot x = b$ has no solution. □

Basic Solutions of Polyhedra in Standard Form

Let $A \in \mathbb{R}^{m\times n}$ with $\text{rank}(A) = m$, $b \in \mathbb{R}^m$, and

$$P = \left\{ x \in \mathbb{R}^n \mid A \cdot x = b,\ x \geq 0 \right\}$$

Theorem 3.37 A point $x \in \mathbb{R}^n$ is a basic solution of P if and only if $A \cdot x = b$ and there are indices $B(1), \dots, B(m) \in \{1, \dots, n\}$ such that

- columns $A_{B(1)}, \dots, A_{B(m)}$ of matrix A are linearly independent, and
- $x_i = 0$ for all $i \notin \{B(1), \dots, B(m)\}$.

Remarks

- $x_{B(1)}, \dots, x_{B(m)}$ are basic variables, the remaining variables non-basic
- the vector of basic variables is denoted by $x_B := (x_{B(1)}, \dots, x_{B(m)})^\top$
- $A_{B(1)}, \dots, A_{B(m)}$ are basic columns of A and form a basis of $\mathbb{R}^m$
- matrix $A_B := (A_{B(1)}, \dots, A_{B(m)}) \in \mathbb{R}^{m\times m}$ is called basis matrix

Proof of Theorem 3.37

"$\Longrightarrow$"

- let $x \in \mathbb{R}^n$ be a basic solution of P, $Ax = b$ is clear by definition.
- there are n linearly independent constraints active at x, i.e., $Ax = b$ and $x_i = 0$ for some $N \subset \{1, \ldots, n\}$ with $|N| = n - m$
- w.l.o.g., $N = \{m + 1, \ldots, n\}$

$$\left[\begin{array}{ccc|c} | & & | & \\ A_{B(1)} & \ldots & A_{B(m)} & * \\ | & & | & \\ \hline & 0 & & I_{n-m} \end{array}\right] \left[\begin{array}{c} | \\ x \\ | \end{array}\right] = \left[\begin{array}{c} | \\ b \\ | \\ \hline | \\ 0 \\ | \end{array}\right] \tag{3.1}$$

- block diagonal matrix has determinant $\det([A_{B(1)}, \ldots, A_{B(m)}]) \det(I_{n-m}) = \det([A_{B(1)}, \ldots, A_{B(m)}])$
- implies $A_{B(1)}, \ldots, A_{B(m)}$ linearly independent

"$\Longleftarrow$"

- active constraints yield matrix as in (3.1)

Corollary: Carathéodory's Theorem

Theorem 3.38 (Carathéodory, 1911) For $S \subseteq \mathbb{R}^n$, every element of $\operatorname{conv}(S)$ can be written as a convex combination of at most $n+1$ points in S.

Proof: Consider $x \in \operatorname{conv}(S)$. Then x can be written as

$$x = \sum_{i=1}^{k} \lambda_i' \, y_i \qquad \text{with } y_1, \ldots, y_k \in S \text{ and } \sum_{i=1}^{k} \lambda_i' = 1, \lambda' \geq 0.$$

Consider the following polyhedron in standard form:

$$P = \left\{ \lambda \in \mathbb{R}^k \;\middle|\; \underbrace{\sum_{i=1}^{k} \lambda_i \, y_i = x, \;\; \sum_{i=1}^{k} \lambda_i = 1,}_{n+1 \text{ equality constraints}} \; \lambda \geq 0 \right\}.$$

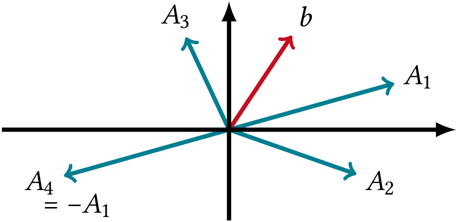
A basic feasible solution $\lambda^* \in P$ yields the desired representation of x. $\square$

Basic Columns and Basic Solutions

Observation 3.39 Let $x \in \mathbb{R}^n$ be a basic solution, then:

- $A_B \cdot x_B = b$ and thus $x_B = A_B^{-1} \cdot b$;
- x is a basic feasible solution if and only if $x_B = A_B^{-1} \cdot b \geq 0$.

Example: $m = 2$



- A_1, A_3 or A_2, A_3 form bases with corresp. basic feasible solutions.
- A_1, A_4 do not form a basis.
- A_1, A_2 and A_2, A_4 and A_3, A_4 form bases with infeasible basic solution.

Bases and Basic Solutions

Corollary 3.40

- Every basis $B(1), \ldots, B(m)$ determines a unique basic solution.
- Thus, different basic solutions correspond to different bases.
- But: two different bases might yield the same basic solution.

Example: If $b = 0$, then $x = 0$ is the only basic solution.

Adjacent Bases

Definition 3.41 Two bases $B(1), \ldots, B(m)$ and $B'(1), \ldots, B'(m)$ are adjacent if $|\{B(1), \ldots, B(m)\} \cap \{B'(1), \ldots, B'(m)\}| = m - 1$.

Observation 3.42

a. Two adjacent basic solutions can always be obtained from two adjacent bases.
b. If two adjacent bases lead to distinct basic solutions, then the latter are adjacent.

3 Geometry of Linear Programming

3.7 Degeneracy

Degeneracy

Definition 3.43 A basic solution x of a polyhedron $P \subseteq \mathbb{R}^n$ is degenerate if more than n constraints are active at x.

Observation 3.44 Let $P = \{x \in \mathbb{R}^n \mid A \cdot x = b,\ x \geq 0\}$ be a polyhedron in standard form with $A \in \mathbb{R}^{m \times n}$, $\operatorname{rank}(A) = m$, and $b \in \mathbb{R}^m$.

a) A basic solution $x \in P$ is degenerate if and only if more than $n - m$ components of x are zero.

b) For a non-degenerate basic solution $x \in P$, there is a unique basis.

Three Different Reasons for Degeneracy

1. redundant variables

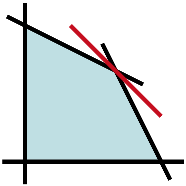
   Example:

$$\begin{aligned} x_1 + x_2 \quad &= 1 \\ x_3 &= 0 \\ x_1,\ x_2,\ x_3 &\geq 0 \end{aligned} \quad \longleftrightarrow \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. redundant constraints

   Example:

$$\begin{aligned} x_1 + 2\,x_2 &\leq 3 \\ 2\,x_1 + x_2 &\leq 3 \\ x_1 + x_2 &\leq 2 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$



3. geometric reasons (non-simple polyhedra)

   Example: Octahedron