6 Simplex: Anticycling and Phase I

6.1 Cycling

Cycling

Problem: If an LP is degenerate, the simplex method might end up in an infinite loop (cycling).

Example:

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	−3/4	20	−1/2	6	0	0	0
$x_5 =$	0	1/4	−8	−1	9	1	0	0
$x_6 =$	0	1/2	−12	−1/2	3	0	1	0
$x_7 =$	1	0	0	1	0	0	0	1

Pivoting rules in this example

- Column selection: steepest descent rule,
  i.e., let non-basic variable with minimal reduced cost $\bar{c}_j < 0$ enter the basis
- Row selection: smallest subscript rule,
  i.e., among basic variables eligible to exit the basis, select the one with smallest subscript

Iteration 1

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\frac{x_{B(i)}}{u_i}$
	3	−3/4	20	−1/2	6	0	0	0	
$x_5 =$	0	1/4	−8	−1	9	1	0	0	0
$x_6 =$	0	1/2	−12	−1/2	3	0	1	0	0
$x_7 =$	1	0	0	1	0	0	0	1	–

Basis change: x_1 enters the basis, x_5 leaves the basis.

Bases visited

$(5, 6, 7)$

Iteration 1

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	$-3/4$	20	$-1/2$	6	0	0	0
$x_5 =$	0	$1/4$	-8	-1	9	1	0	0
$x_6 =$	0	$1/2$	-12	$-1/2$	3	0	1	0
$x_7 =$	1	0	0	1	0	0	0	1

Basis change: x_1 enters the basis, x_5 leaves the basis.

Bases visited

$(5, 6, 7)$

Iteration 1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
3	0	−4	−7/2	33	3	0	0
$x_5 =$ 0	1/4	−8	−1	9	1	0	0
$x_6 =$ 0	1/2	−12	−1/2	3	0	1	0
$x_7 =$ 1	0	0	1	0	0	0	1

Basis change: x_1 enters the basis, x_5 leaves the basis.

Bases visited

$(5, 6, 7)$

Iteration 1

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	0	−4	−7/2	33	3	0	0
$x_5 =$	0	1/4	−8	−1	9	1	0	0
$x_6 =$	0	1/2	−12	−1/2	3	0	1	0
$x_7 =$	1	0	0	1	0	0	0	1

Basis change: x_1 enters the basis, x_5 leaves the basis.

Bases visited

$(5, 6, 7)$

Iteration 1

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	0	−4	−7/2	33	3	0	0
$x_5 =$	0	1/4	−8	−1	9	1	0	0
$x_6 =$	0	0	4	3/2	−15	−2	1	0
$x_7 =$	1	0	0	1	0	0	0	1

Basis change: x_1 enters the basis, x_5 leaves the basis.

Bases visited

(5, 6, 7)

Iteration 1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
3	0	−4	−7/2	33	3	0	0
$x_5 =$ 0	1/4	−8	−1	9	1	0	0
$x_6 =$ 0	0	4	3/2	−15	−2	1	0
$x_7 =$ 1	0	0	1	0	0	0	1

Basis change: x_1 enters the basis, x_5 leaves the basis.

Bases visited

$(5, 6, 7)$

Iteration 1

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	0	−4	−7/2	33	3	0	0
$x_5 =$	0	1/4	−8	−1	9	1	0	0
$x_6 =$	0	0	4	3/2	−15	−2	1	0
$x_7 =$	1	0	0	1	0	0	0	1

Basis change: x_1 enters the basis, x_5 leaves the basis.

Bases visited

$(5, 6, 7)$

Iteration 1

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	0	-4	$-7/2$	33	3	0	0
$x_1 =$	0	1	-32	-4	36	4	0	0
$x_6 =$	0	0	4	$3/2$	-15	-2	1	0
$x_7 =$	1	0	0	1	0	0	0	1

Basis change: x_1 enters the basis, x_5 leaves the basis.

Bases visited

$(5, 6, 7)$

Iteration 2

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\frac{x_{B(i)}}{u_i}$
	3	0	−4	−7/2	33	3	0	0	
$x_1 =$	0	1	−32	−4	36	4	0	0	–
$x_6 =$	0	0	4	3/2	−15	−2	1	0	0
$x_7 =$	1	0	0	1	0	0	0	1	–

Basis change: x_2 enters the basis, x_6 leaves the basis.

Bases visited

$(5, 6, 7) \longrightarrow (1, 6, 7)$

Iteration 3

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\frac{x_{B(i)}}{u_i}$
	3	0	0	−2	18	1	1	0	
$x_1 =$	0	1	0	8	−84	−12	8	0	0
$x_2 =$	0	0	1	3/8	−15/4	−1/2	1/4	0	0
$x_7 =$	1	0	0	1	0	0	0	1	1

Basis change: x_3 enters the basis, x_1 leaves the basis.

Bases visited

$(5, 6, 7) \longrightarrow (1, 6, 7) \longrightarrow (1, 2, 7)$

Iteration 4

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\frac{x_{B(i)}}{u_i}$
	3	1/4	0	0	−3	−2	3	0	
$x_3 =$	0	1/8	0	1	−21/2	−3/2	1	0	–
$x_2 =$	0	−3/64	1	0	3/16	1/16	−1/8	0	0
$x_7 =$	1	−1/8	0	0	21/2	3/2	−1	1	2/21

Basis change: x_4 enters the basis, x_2 leaves the basis.

Bases visited

$(5, 6, 7) \longrightarrow (1, 6, 7) \longrightarrow (1, 2, 7) \longrightarrow (3, 2, 7)$

Iteration 5

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\frac{x_{B(i)}}{u_i}$
	3	−1/2	16	0	0	−1	1	0	
$x_3 =$	0	−5/2	56	1	0	2	−6	0	0
$x_4 =$	0	−1/4	16/3	0	1	1/3	−2/3	0	0
$x_7 =$	1	5/2	−56	0	0	−2	6	1	−

Basis change: x_5 enters the basis, x_3 leaves the basis.

Bases visited

$(5,6,7) \longrightarrow (1,6,7) \longrightarrow (1,2,7) \longrightarrow (3,2,7) \longrightarrow (3,4,7)$

Observation

After 4 pivoting iterations our basic feasible solution still has not changed.

Iteration 6

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\frac{x_{B(i)}}{u_i}$
	3	−7/4	44	1/2	0	0	−2	0	
$x_5 =$	0	−5/4	28	1/2	0	1	−3	0	–
$x_4 =$	0	1/6	−4	−1/6	1	0	1/3	0	0
$x_7 =$	1	0	0	1	0	0	0	1	–

Basis change: x_6 enters the basis, x_4 leaves the basis.

Bases visited

$(5, 6, 7) \longrightarrow (1, 6, 7) \longrightarrow (1, 2, 7) \longrightarrow (3, 2, 7) \longrightarrow (3, 4, 7)$
$\longrightarrow (5, 4, 7)$

Back at the Beginning

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	$-3/4$	20	$-1/2$	6	0	0	0
$x_5 =$	0	$1/4$	-8	-1	9	1	0	0
$x_6 =$	0	$1/2$	-12	$-1/2$	3	0	1	0
$x_7 =$	1	0	0	1	0	0	0	1

Bases visited

$(5,6,7) \longrightarrow (1,6,7) \longrightarrow (1,2,7) \longrightarrow (3,2,7) \longrightarrow (3,4,7)$
$\longrightarrow (5,4,7) \longrightarrow (5,6,7)$

This is the same basis that we started with.

Conclusion

Continuing with the pivoting rules we agreed on at the beginning, the simplex method will never terminate in this example.

Introduction to

Linear and Combinatorial Optimization

6 Simplex: Anticycling and Phase I

6.2 Lexicographic & Bland's Rule

Definition 6.1

- a A vector $u \in \mathbb{R}^n$ is lexicographically positive (negative) if $u \neq 0$ and the first nonzero entry of u is positive (negative). Symbolically, we write $u >_L 0$ (resp. $u <_L 0$).
- b A vector $u \in \mathbb{R}^n$ is lexicographically larger (smaller) than a vector $v \in \mathbb{R}^n$ if $u - v >_L 0$ (resp. $u - v <_L 0$).We write $u >_L v$ (resp. $u <_L v$).

Examples:

$$(0, 2, 3, 0)^\top >_L (0, 2, 1, 4)^\top$$
$$(0, 4, 5, 0)^\top <_L (1, 2, 1, 2)^\top$$

Lexicographic Pivoting Rule

Lexicographic pivoting rule in the full tableau implementation

i Choose an arbitrary column A_j with $\bar{c}_j < 0$ to enter the basis. Let $u := A_B^{-1} A_j$ be the jth column of the tableau.

ii For each i with $u_i > 0$, divide the ith row of the tableau by u_i and choose the lexicographically smallest row ℓ. Then the ℓth basic variable $x_{B(\ell)}$ exits the basis.

Remarks.

- The lexicographic pivoting rule always leads to a unique choice for the exiting variable. Otherwise two rows of $A_B^{-1} A$ would have to be linearly dependent which contradicts our assumption on the matrix A.
- The chosen ℓ meets the requirement

$$\frac{x_{B(\ell)}}{u_\ell} = \min_{i\,:\,u_i>0} \frac{x_{B(i)}}{u_i}$$

because after dividing the ith row by u_i its zeroth entry is $\frac{x_{B(i)}}{u_i}$.

Lexicographic Pivoting Rule (Cont.)

Theorem 6.2 Suppose that the simplex method starts with lexicographically positive rows $1, \ldots, m$ in the simplex tableau. Suppose that the lexicographic pivoting rule is followed. Then:

a. All rows remain lexicographically positive throughout the algorithm.

b. The zeroth row strictly increases lexicographically at each iteration.

c. The simplex method terminates after a finite number of iterations.

Corollary 6.3 Every bounded LP in standard form has an optimal basis matrix.

Proof of Theorem 6.2

Proof: a all rows remain lexicographically positive

- let $T = A_B^{-1}[b, A]$ be the full tableau, assume all rows $T_{i,.} >_L 0$
- let $\bar{B} = (B(1), \dots, B(\ell - 1), j, B(\ell), B(m))$
- then $u = A_B^{-1} A_j$, $u_\ell > 0$, and $\frac{T_{\ell,.}}{u_\ell} <_L \frac{T_{i,.}}{u_i}$ for all i with $u_i > 0$
- let $\bar{T} = A_{\bar{B}}^{-1}[b, A]$
- $\bar{T}_{\ell,.} = \frac{1}{u_\ell} T_{\ell,.} >_L 0$
- for $i \neq \ell$, $\bar{T}_{i,.} = T_{i,.} - \frac{u_i}{u_\ell} T_{\ell,.}$
  - if $u_i \leq 0$, then $\bar{T}_{i,.}$ is the sum of of a lex. pos. row and a row that is lex. pos. or zero, so it is lex. pos.
  - if $u_i > 0$, then $\frac{T_{i,.}}{u_i} >_L \frac{T_{\ell,.}}{u_\ell} \implies \bar{T}_{i,.} = T_{i,.} - \frac{u_i}{u_\ell} T_{\ell,.} >_L 0$

b zeroth row strictly increases lexicographically

- $\bar{T}_{0,.} = T_{0,.} - \frac{\bar{c}_j}{u_\ell} T_{\ell,.} \implies \bar{T}_{0,.} >_L T_{0,.}$ since $\frac{\bar{c}_j}{u_\ell} < 0$ and $T_{\ell,.} >_L 0$

c simplex method terminates

- follows from b since zeroth row is uniquely determined by current basis

Lexicographic Pivoting Rule: An Example

Example:

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	$-3/4$	20	$-1/2$	6	0	0	0
$x_5 =$	0	$\frac{1}{4}$	-8	-1	9	1	0	0
$x_6 =$	0	$\frac{1}{2}$	-12	$-\frac{1}{2}$	3	0	1	0
$x_7 =$	1	0	0	1	0	0	0	1

Pivoting rules in this example

- Column selection: steppest descent rule,
  i.e., let non-basic variable with minimal reduced cost $\bar{c}_j < 0$ enter the basis
- Row selection: lexicographic rule,
  i.e., among basic variables eligible to exit the basis, select the one lexicographically minimizing $T_{i,.}/u_i$

Iteration 1

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	$-\frac{3}{4}$	20	$-\frac{1}{2}$	6	0	0	0
$x_5 =$	0	$\frac{1}{4}$	−8	−1	9	1	0	0
$x_6 =$	0	$\frac{1}{2}$	−12	$-\frac{1}{2}$	3	0	1	0
$x_7 =$	1	0	0	1	0	0	0	1

$T_{i,.}/u_i$ for i with $u_i > 0$:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	1	−32	−4	36	4	0	0
0	1	−24	−1	6	0	2	0

Basis change: x_1 enters the basis, x_5 leaves the basis.

Iteration 2

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	0	-4	$-\frac{7}{2}$	33	3	0	0
$x_1 =$	0	1	-32	-4	36	4	0	0
$x_6 =$	0	0	4	$\frac{3}{2}$	-15	-2	1	0
$x_7 =$	1	0	0	1	0	0	0	1

Basis change: x_2 enters the basis, x_6 leaves the basis.

Iteration 3

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	0	0	-2	18	1	1	0
$x_1 =$	0	1	0	8	-84	-12	8	0
$x_2 =$	0	0	1	$\frac{3}{8}$	$-\frac{15}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	0
$x_7 =$	1	0	0	1	0	0	0	1

$T_{i,.}/u_i$ for i with $u_i > 0$:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	$\frac{1}{8}$	0	1	$-\frac{21}{2}$	$-\frac{3}{2}$	1	0
0	0	$\frac{8}{3}$	1	-10	$-\frac{4}{3}$	$\frac{2}{3}$	0
1	0	0	1	0	0	0	1

Basis change: x_3 enters the basis, x_2 leaves the basis.

Iteration 4

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	0	$\frac{16}{3}$	0	-2	$-\frac{5}{3}$	$\frac{7}{3}$	0
$x_1 =$	0	1	$-\frac{64}{3}$	0	-4	$-\frac{4}{3}$	$\frac{8}{3}$	0
$x_3 =$	0	0	$\frac{8}{3}$	1	-10	$-\frac{4}{3}$	$\frac{2}{3}$	0
$x_7 =$	1	0	$-\frac{8}{3}$	0	10	$\frac{4}{3}$	$-\frac{2}{3}$	1

Basis change: x_4 enters the basis, x_7 leaves the basis.

Iteration 5

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	$\frac{16}{5}$	0	$\frac{24}{5}$	0	0	$-\frac{7}{5}$	$\frac{11}{5}$	$\frac{1}{5}$
$x_1 =$	$\frac{2}{5}$	1	$-\frac{112}{5}$	0	0	$-\frac{4}{5}$	$\frac{12}{5}$	$\frac{2}{5}$
$x_3 =$	1	0	0	1	0	0	0	1
$x_4 =$	$\frac{1}{10}$	0	$-\frac{4}{15}$	0	1	$\frac{2}{15}$	$-\frac{1}{15}$	$\frac{1}{10}$

Basis change: x_5 enters the basis, x_4 leaves the basis.

Finally at Optimal Basis

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	$\frac{17}{4}$	0	2	0	$\frac{21}{2}$	0	$\frac{3}{2}$	$\frac{5}{4}$
$x_1 =$	1	1	-24	0	6	0	2	1
$x_3 =$	1	0	0	1	0	0	0	1
$x_5 =$	$\frac{3}{4}$	0	-2	0	$\frac{15}{2}$	1	$-\frac{1}{2}$	$\frac{3}{4}$

Thus $(1, 0, 1, 0, 3/4, 0, 0)$ is an optimal solution with cost $-17/4$.

Remarks on Lexicographic Pivoting Rule

- Lexicographic rule can be derived by considering small perturbation of right hand side b leading to non-degenerate problem (see exercises).

- Lexicographic pivoting rule can also be used in conjunction with revised simplex method, provided that A_B^{-1} is computed explicitly
(not the case in sophisticated implementations).

- Assumption in theorem on lexicographically positive rows in tableau can be made without loss of generality:
    - Rearrange columns of A such that basic columns (forming identity matrix in tableau) come first.
    - Since zeroth column is nonnegative for basic feasible solution, all rows are lexicographically positive.

Bland's Rule

Smallest subscript pivoting rule (Bland's rule)

i. Choose the column A_j with $\bar{c}_j < 0$ and j minimal to enter the basis.
ii. Among all basic variables x_i that could exit the basis, select the one with smallest i.

Theorem 6.4 Simplex method with Bland's rule terminates after finitely many iterations.

Proof: see exercise

Remark

Bland's rule is compatible with an implementation of the revised simplex method in which the reduced costs of the nonbasic variables are computed one at a time, in the natural order, until a negative one is discovered.

6 Simplex: Anticycling and Phase I

6.3 Phase I

Introducing Artificial Variables

- how do we obtain an initial feasible solution?

Example:

$$
\begin{array}{lrcrcrcrcl}
\min & x_1 & + & x_2 & + & x_3 & & & & \\
\text{s.t.} & x_1 & + & 2\,x_2 & + & 3\,x_3 & & & = & 3 \\
 & -x_1 & + & 2\,x_2 & + & 6\,x_3 & & & = & 2 \\
 & & & 4\,x_2 & + & 9\,x_3 & & & = & 5 \\
 & & & & & 3\,x_3 & + & x_4 & = & 1 \\
 & & & & & & & x_1, \ldots, x_4 & \geq & 0
\end{array}
$$

Auxiliary problem with artificial variables:

$$
\begin{array}{lrrrrrrrrcl}
\min & & & & & x_5 & +x_6 & +x_7 & +x_8 & & \\
\text{s.t.} & x_1 & +2\,x_2 & +3\,x_3 & & +x_5 & & & & = & 3 \\
 & -x_1 & +2\,x_2 & +6\,x_3 & & & +x_6 & & & = & 2 \\
 & & 4\,x_2 & +9\,x_3 & & & & +x_7 & & = & 5 \\
 & & & 3\,x_3 & +x_4 & & & & +x_8 & = & 1 \\
 & & & & & & & & x_1, \ldots, x_4, x_5, \ldots, x_8 & \geq & 0
\end{array}
$$

Auxiliary Problem

Auxiliary problem with artificial variables:

$$
\begin{array}{rrrrrrrrrrcl}
\min & & & & & x_5 & +x_6 & +x_7 & +x_8 & & & \\
\text{s.t.} & x_1 & +2\,x_2 & +3\,x_3 & & +x_5 & & & & = & 3 \\
 & -x_1 & +2\,x_2 & +6\,x_3 & & & +x_6 & & & = & 2 \\
 & & 4\,x_2 & +9\,x_3 & & & & +x_7 & & = & 5 \\
 & & & 3\,x_3 & +x_4 & & & & +x_8 & = & 1 \\
 & & & & & \multicolumn{4}{r}{x_1, \ldots, x_4, x_5, \ldots, x_8} & \geq & 0
\end{array}
$$

Observation

$x = (0, 0, 0, 0, 3, 2, 5, 1)$ is a basic feasible solution for this problem with basic variables (x_5, x_6, x_7, x_8). We can thus form the initial tableau.

Initial Tableau

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	0	0	0	0	0	1	1	1	1
$x_5 =$	3	1	2	3	0	1	0	0	0
$x_6 =$	2	−1	2	6	0	0	1	0	0
$x_7 =$	5	0	4	9	0	0	0	1	0
$x_8 =$	1	0	0	3	1	0	0	0	1

Calculate reduced costs by eliminating the nonzero-entries for the basis-variables, this works since $\bar{c} = c - c_B A_B^{-1} A = -\mathbf{1}A$

Now we can proceed as seen before...

Initial Tableau

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	0	0	0	0	0	1	1	1	1
$x_5 =$	3	1	2	3	0	1	0	0	0
$x_6 =$	2	-1	2	6	0	0	1	0	0
$x_7 =$	5	0	4	9	0	0	0	1	0
$x_8 =$	1	0	0	3	1	0	0	0	1

Calculate reduced costs by eliminating the nonzero-entries for the basis-variables, this works since $\bar{c} = c - c_B A_B^{-1} A = -\mathbf{1}A$

Now we can proceed as seen before...

Initial Tableau

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	-3	-1	-2	-3	0	0	1	1	1
$x_5 =$	3	1	2	3	0	1	0	0	0
$x_6 =$	2	-1	2	6	0	0	1	0	0
$x_7 =$	5	0	4	9	0	0	0	1	0
$x_8 =$	1	0	0	3	1	0	0	0	1

Calculate reduced costs by eliminating the nonzero-entries for the basis-variables, this works since $\bar{c} = c - c_B A_B^{-1} A = -\mathbf{1}A$

Now we can proceed as seen before...

Initial Tableau

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	-5	0	-4	-9	0	0	0	1	1
$x_5 =$	3	1	2	3	0	1	0	0	0
$x_6 =$	2	-1	2	6	0	0	1	0	0
$x_7 =$	5	0	4	9	0	0	0	1	0
$x_8 =$	1	0	0	3	1	0	0	0	1

Calculate reduced costs by eliminating the nonzero-entries for the basis-variables, this works since $\bar{c} = c - c_B A_B^{-1} A = -\mathbf{1}A$

Now we can proceed as seen before...

Initial Tableau

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
-10	0	-8	-18	0	0	0	0	1
$x_5 =$ 3	1	2	3	0	1	0	0	0
$x_6 =$ 2	-1	2	6	0	0	1	0	0
$x_7 =$ 5	0	4	9	0	0	0	1	0
$x_8 =$ 1	0	0	3	1	0	0	0	1

Calculate reduced costs by eliminating the nonzero-entries for the basis-variables, this works since $\bar{c} = c - c_B A_B^{-1} A = -\mathbf{1} A$

Now we can proceed as seen before...

Initial Tableau

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	-11	0	-8	-21	-1	0	0	0	0
$x_5 =$	3	1	2	3	0	1	0	0	0
$x_6 =$	2	-1	2	6	0	0	1	0	0
$x_7 =$	5	0	4	9	0	0	0	1	0
$x_8 =$	1	0	0	3	1	0	0	0	1

Calculate reduced costs by eliminating the nonzero-entries for the basis-variables, this works since $\bar{c} = c - c_B A_B^{-1} A = -\mathbf{1}A$

Now we can proceed as seen before...

Minimizing the Auxiliary Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	-11	0	-8	-21	-1	0	0	0	0
$x_5 =$	3	1	2	3	0	1	0	0	0
$x_6 =$	2	-1	2	6	0	0	1	0	0
$x_7 =$	5	0	4	9	0	0	0	1	0
$x_8 =$	1	0	0	3	1	0	0	0	1

Basis change: x_4 enters the basis, x_8 exits.

Minimizing the Auxiliary Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	-10	0	-8	-18	0	0	0	0	1
$x_5 =$	3	1	2	3	0	1	0	0	0
$x_6 =$	2	-1	2	6	0	0	1	0	0
$x_7 =$	5	0	4	9	0	0	0	1	0
$x_4 =$	1	0	0	3	1	0	0	0	1

Basis change: x_3 enters the basis, x_4 exits.

Minimizing the Auxiliary Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	-4	0	-8	0	6	0	0	0	7
$x_5 =$	2	1	2	0	-1	1	0	0	-1
$x_6 =$	0	-1	2	0	-2	0	1	0	-2
$x_7 =$	2	0	4	0	-3	0	0	1	-3
$x_3 =$	1/3	0	0	1	1/3	0	0	0	1/3

Basis change: x_2 enters the basis, x_6 exits.

Minimizing the Auxiliary Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	-4	-4	0	0	-2	0	4	0	-1
$x_5 =$	2	2	0	0	1	1	-1	0	1
$x_2 =$	0	$-1/2$	1	0	-1	0	$1/2$	0	-1
$x_7 =$	2	2	0	0	1	0	-2	1	1
$x_3 =$	$1/3$	0	0	1	$1/3$	0	0	0	$1/3$

Basis change: x_1 enters the basis, x_5 exits.

Minimizing the Auxiliary Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	0	0	0	0	0	2	2	0	1
$x_1 =$	1	1	0	0	1/2	1/2	−1/2	0	1/2
$x_2 =$	1/2	0	1	0	−3/4	1/4	1/4	0	−3/4
$x_7 =$	0	0	0	0	0	−1	−1	1	0
$x_3 =$	1/3	0	0	1	1/3	0	0	0	1/3

Basic feasible solution for auxiliary problem with (auxiliary) cost value 0

$\Longrightarrow$ Also feasible for the original problem - but not (yet) basic.

Obtaining a Basis for the Original Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	0	0	0	0	0	2	2	0	1
$x_1 =$	1	1	0	0	1/2	1/2	−1/2	0	1/2
$x_2 =$	1/2	0	1	0	−3/4	1/4	1/4	0	−3/4
$x_7 =$	0	0	0	0	0	−1	−1	1	0
$x_3 =$	1/3	0	0	1	1/3	0	0	0	1/3

Observation

Restricting the tableau to the original variables, we get a zero-row.

Thus the original equations are linearily dependent.

$\longrightarrow$ We can remove the third row.

Obtaining a Basis for the Original Problem

		x_1	x_2	x_3	x_4
	$-11/6$	0	0	0	$-1/12$
$x_1 =$	1	1	0	0	$1/2$
$x_2 =$	$1/2$	0	1	0	$-3/4$
$x_3 =$	$1/3$	0	0	1	$1/3$

We finally obtain a basic feasible solution for the original problem.

After computing the reduced costs for this basis (as seen in the beginning), the simplex method can start with its typical iterations.

Omitting Artificial Variables

Auxiliary problem

$$
\begin{array}{lrrrrrrrrrrr}
\min & & & & & x_5 & +x_6 & +x_7 & +x_8 & & \\
\text{s.t.} & x_1 & +2\,x_2 & +3\,x_3 & & +x_5 & & & & = & 3 \\
& -x_1 & +2\,x_2 & +6\,x_3 & & & +x_6 & & & = & 2 \\
& & 4\,x_2 & +9\,x_3 & & & & +x_7 & & = & 5 \\
& & & 3\,x_3 & +x_4 & & & & +x_8 & = & 1 \\
& & & & & & & & x_1,\dots,x_8 & \geq & 0
\end{array}
$$

Artificial variable x_8 could have been omitted by setting x_4 to 1 in the initial basis. This is possible as x_4 does only appear in one constraint.

Generally, this can be done, e.g., with all slack variables that have nonnegative right hand sides.

Phase I of the Simplex Method

Given: LP in standard form: $\min\{c^\top \cdot x \mid A \cdot x = b,\ x \geq 0\}$

i Transform problem such that $b \geq 0$ (multiply constraints by -1).

ii Introduce artificial variables $y_1, \ldots, y_m$ and solve auxiliary problem

$$\min \sum_{i=1}^{m} y_i \quad \text{s.t. } A \cdot x + I_m \cdot y = b,\ x, y \geq 0.$$

iii If optimal cost is positive, then STOP (original LP is infeasible).

iv If no artificial variable is in final basis, eliminate artificial variables and columns and STOP (feasible basis for original LP has been found).

v If ℓth basic variable is artificial, find $j \in \{1, \ldots, n\}$ with ℓth entry in $A_B^{-1} \cdot A_j$ nonzero. Use this entry as pivot element and replace ℓth basic variable with x_j.

vi If no such $j \in \{1, \ldots, n\}$ exists, eliminate ℓth row (constraint).

Two-phase simplex method

1. Given an LP in standard from, first run phase I.
2. If phase I yields a basic feasible solution for the original LP, enter "phase II" (see above).

Possible outcomes of the two-phase simplex method

i. Problem is infeasible (detected in phase I).
ii. Problem is feasible but rows of A are linearly dependent (detected and corrected at the end of phase I by eliminating redundant constraints.)
iii. Optimal cost is $-\infty$ (detected in phase II).
iv. Problem has optimal basic feasible solution (found in phase II).

Remark: ii is not an outcome but only an intermediate result leading to outcome iii or iv.

Big-M Method

Alternative idea: Combine the two phases into one by introducing sufficiently large penalty costs for artificial variables.

This way, the LP

$$\begin{array}{rrcl} \min & \sum_{i=1}^{n} c_i x_i & & \\ \text{s.t.} & A \cdot x & = & b \\ & x & \geq & 0 \end{array}$$

becomes:

$$\begin{array}{rrcrcl} \min & \sum_{i=1}^{n} c_i x_i & + & M \cdot \sum_{j=1}^{m} y_j & & \\ \text{s.t.} & A \cdot x & + & I_m \cdot y & = & b \\ & & & x, y & \geq & 0 \end{array}$$

Remark: If M is sufficiently large and the original program has a feasible solution, all artificial variables will be driven to zero by the simplex method.

How to Choose M?

Observation

Initially, M only occurs in the zeroth row. As the zeroth row never becomes pivot row, this property is maintained while the simplex method is running.

All we need to have is an order on all values that can appear as reduced cost coefficients.

Order on cost coefficients

$$aM + b < cM + d \quad :\Longleftrightarrow \quad (a < c) \vee (a = c \wedge b < d)$$

In particular, $-aM + b < 0 < aM + b$ for any positive a and arbitrary b, and we can decide whether a cost coefficient is negative or not.

$\longrightarrow$ There is no need to give M a fixed numerical value.

Example

Example:

$$
\begin{array}{rrrrrrrrrr}
\min & x_1 & + & x_2 & + & x_3 & & & & \\
\text{s.t.} & x_1 & + & 2\,x_2 & + & 3\,x_3 & & & = & 3 \\
& -x_1 & + & 2\,x_2 & + & 6\,x_3 & & & = & 2 \\
& & & 4\,x_2 & + & 9\,x_3 & & & = & 5 \\
& & & & & 3\,x_3 & + & x_4 & = & 1 \\
& & & & & & & x_1, \ldots, x_4 & \geq & 0
\end{array}
$$

Introducing Artificial Variables and M

Auxiliary problem:

$$
\begin{array}{lrrrrrrrcl}
\min & x_1 & +x_2 & +x_3 & & +M\,x_5 & +M\,x_6 & +M\,x_7 & & \\
\text{s.t.} & x_1 & +2\,x_2 & +3\,x_3 & & x_5 & & & = & 3 \\
 & -x_1 & +2\,x_2 & +6\,x_3 & & & +x_6 & & = & 2 \\
 & & 4\,x_2 & +9\,x_3 & & & & +x_7 & = & 5 \\
 & & & 3\,x_3 & +x_4 & & & & = & 1 \\
 & & & & & & & x_1, \ldots, x_4, x_5, x_6, x_7 & \geq & 0
\end{array}
$$

Note that this time the unnecessary artificial variable x_8 has been omitted.

We start off with $(x_5, x_6, x_7, x_4) = (3, 2, 5, 1)$.

Forming the Initial Tableau

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	1	1	1	0	M	M	M
3	1	2	3	0	1	0	0
2	−1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

Compute reduced costs by eliminating the nonzero entries for the basic variables.

Forming the Initial Tableau

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	1	1	1	0	M	M	M
3	1	2	3	0	1	0	0
2	−1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

Compute reduced costs by eliminating the nonzero entries for the basic variables.

Forming the Initial Tableau

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-3M$	$-M+1$	$-2M+1$	$-3M+1$	0	0	M	M
3	1	2	3	0	1	0	0
2	−1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

Compute reduced costs by eliminating the nonzero entries for the basic variables.

Forming the Initial Tableau

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-5M$	1	$-4M + 1$	$-9M + 1$	0	0	0	M
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

Compute reduced costs by eliminating the nonzero entries for the basic variables.

Forming the Initial Tableau

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-10M$	1	$-8M + 1$	$-18M + 1$	0	0	0	0
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

Compute reduced costs by eliminating the nonzero entries for the basic variables.

Forming the Initial Tableau

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-10M$	1	$-8M + 1$	$-18M + 1$	0	0	0	0
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

Compute reduced costs by eliminating the nonzero entries for the basic variables.

First Iteration

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-10M$	1	$-8M+1$	$-18M+1$	0	0	0	0
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

Reduced costs for x_2 and x_3 are negative.

Basis change: x_3 enters the basis, x_4 leaves.

Second Iteration

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-4M - 1/3$	1	$-8M + 1$	0	$6M - 1/3$	0	0	0
2	1	2	0	-1	1	0	0
0	-1	2	0	-2	0	1	0
2	0	4	0	-3	0	0	1
1/3	0	0	1	1/3	0	0	0

Basis change: x_2 enters the basis, x_6 leaves.

Third Iteration

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-4M - 1/3$	$-4M + 3/2$	0	0	$-2M + 2/3$	0	$4M - 1/2$	0
2	2	0	0	1	1	-1	0
0	$-1/2$	1	0	-1	0	$1/2$	0
2	2	0	0	1	0	-2	1
$1/3$	0	0	1	$1/3$	0	0	0

Basis change: x_1 enters the basis, x_5 leaves.

Fourth Iteration

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-11/6$	0	0	0	$-1/12$	$2M - 3/4$	$2M + 1/4$	0
1	1	0	0	$1/2$	$1/2$	$-1/2$	0
$1/2$	0	1	0	$-3/4$	$1/4$	$1/4$	0
0	0	0	0	0	-1	-1	1
$1/3$	0	0	1	$1/3$	0	0	0

Note that all artificial variables have already been driven to 0.

Basis change: x_4 enters the basis, x_3 leaves.

Fifth Iteration

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-7/4$	0	0	$1/4$	0	$2M - 3/4$	$2M + 1/4$	0
$1/2$	1	0	$-3/2$	0	$1/2$	$-1/2$	0
$5/4$	0	1	$9/4$	0	$1/4$	$1/4$	0
0	0	0	0	0	-1	-1	1
1	0	0	3	1	0	0	0

We now have an optimal solution to the auxiliary problem, as all costs are nonnegative (M presumed large enough).

By elimiating the third row as in the previous example, we get a basic feasible and also optimal solution to the original problem.