Introduction to

Linear and Combinatorial Optimization

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Duality Theory

7.1 Motivation and Definition

Motivation

For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$, consider the linear program

$$\min\ c^\top \cdot x \qquad \text{s.t.} \quad A \cdot x \geq b,\ x \geq 0$$

Question: How to derive lower bounds on the optimal solution value?

Idea: For $p \in \mathbb{R}^m$ with $p \geq 0$: $\quad A \cdot x \geq b \quad \Longrightarrow \quad (p^\top \cdot A) \cdot x \geq p^\top \cdot b$

Thus, if $c^\top \geq p^\top \cdot A$, then

$$c^\top \cdot x \geq (p^\top \cdot A) \cdot x \geq p^\top \cdot b \qquad \text{for all feasible solutions } x.$$

Find the best (largest) lower bound in this way:

$$\begin{array}{ll} \max\limits_p & p^\top \cdot b \\ \text{s.t.} & p^\top \cdot A \leq c^\top \\ & p \geq 0 \end{array} \qquad \longleftrightarrow \qquad \begin{array}{ll} \max\limits_p & b^\top \cdot p \\ \text{s.t.} & A^\top \cdot p \leq c \\ & p \geq 0 \end{array}$$

This LP is the dual linear program of our initial LP.

Primal and Dual Linear Program

Consider the general linear program:

$$\begin{array}{lll} \min\limits_{x} & c^\top \cdot x & \\ \text{s.t.} & a_i^\top \cdot x \geq b_i & \text{for } i \in M_1 \\ & a_i^\top \cdot x \leq b_i & \text{for } i \in M_2 \\ & a_i^\top \cdot x = b_i & \text{for } i \in M_3 \\ & x_j \geq 0 & \text{for } j \in N_1 \\ & x_j \leq 0 & \text{for } j \in N_2 \\ & x_j \text{ free} & \text{for } j \in N_3 \end{array}$$

Obtain a lower bound:

$$\begin{array}{lll} \max\limits_{p} & p^\top \cdot b & \\ \text{s.t.} & p_i \geq 0 & \text{for } i \in M_1 \\ & p_i \leq 0 & \text{for } i \in M_2 \\ & p_i \text{ free} & \text{for } i \in M_3 \\ & p^\top \cdot A_j \leq c_j & \text{for } j \in N_1 \\ & p^\top \cdot A_j \geq c_j & \text{for } j \in N_2 \\ & p^\top \cdot A_j = c_j & \text{for } j \in N_3 \end{array}$$

The linear program on the right hand side is the dual linear program of the primal linear program on the left hand side.

Primal and Dual Variables and Constraints

primal LP (minimize)		dual LP (maximize)	
constraints	$\geq b_i$	≥ 0	variables
	$\leq b_i$	≤ 0	
	$= b_i$	free	
variables	≥ 0	$\leq c_i$	constraints
	≤ 0	$\geq c_i$	
	free	$= c_i$	

Examples of Primal and Dual LPs

primal LP	dual LP
$\min\limits_{x} \quad c^\top \cdot x$ s.t. $\quad A \cdot x \geq b$	$\max\limits_{p} \quad p^\top \cdot b$ s.t. $\quad p^\top \cdot A = c^\top$ $p \geq 0$
$\min\limits_{x} \quad c^\top \cdot x$ s.t. $\quad A \cdot x = b$ $x \geq 0$	$\max\limits_{p} \quad p^\top \cdot b$ s.t. $\quad p^\top \cdot A \leq c^\top$

Examples of Primal and Dual LPs

primal LP	dual LP
$\begin{array}{rl} \min\limits_{x} & c^\top \cdot x \\ \text{s.t.} & A \cdot x \geq b \end{array}$	$\begin{array}{rl} \max\limits_{p} & b^\top \cdot p \\ \text{s.t.} & A^\top \cdot p = c \\ & p \geq 0 \end{array}$
$\begin{array}{rl} \min\limits_{x} & c^\top \cdot x \\ \text{s.t.} & A \cdot x = b \\ & x \geq 0 \end{array}$	$\begin{array}{rl} \max\limits_{p} & b^\top \cdot p \\ \text{s.t.} & A^\top \cdot p \leq c \end{array}$

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7.2 Basic Properties

Basic Properties of the Dual

Theorem 7.1 The dual of the dual LP is the primal LP.

Proof:

- for the special case of a primal LP in standard form

$$\begin{tikzcd}
\begin{array}{rl} \min\limits_{x} & c^\top \cdot x \\ \text{s.t.} & A \cdot x = b \\ & x \geq 0 \end{array} \arrow[r, leftrightarrow, "\text{dualize}"] & \begin{array}{rl} \max\limits_{p} & b^\top p \\ \text{s.t.} & A^\top p \leq c \end{array} \arrow[r, Leftrightarrow] & \begin{array}{rl} \min\limits_{p} & -b^\top p \\ \text{s.t.} & A^\top p \leq c \end{array} \arrow[d, leftrightarrow, "\text{dualize}"] \\
\begin{array}{rl} \min\limits_{x} & c^\top \cdot x \\ \text{s.t.} & A \cdot x = b \\ & x \geq 0 \end{array} \arrow[r, Leftrightarrow] & \begin{array}{rl} \max\limits_{x} & -c^\top x \\ \text{s.t.} & -Ax = -b \\ & x \geq 0 \end{array} \arrow[r, Leftrightarrow, "x:=-y"] & \begin{array}{rl} \max\limits_{y} & c^\top y \\ \text{s.t.} & Ay = -b \\ & y \leq 0 \end{array}
\end{tikzcd}$$

□

Basic Properties of the Dual (Cont.)

Theorem 7.2 Let Π_1 and Π_2 be two LPs where Π_2 has been obtained from Π_1 by (several) transformations of the following type:

- i replace a free variable by the difference of two non-negative variables;
- ii replace an inequality by an equation introducing a slack variable;
- iii eliminate a row that is a linear combination of other rows in a feasible equality system.

Then the dual of Π_1 is equivalent to the dual of Π_2.

Proof of Theorem 7.2 (i)

Proof: i replacing free variable by two positive variables

$$\begin{tikzcd}[column sep=huge, row sep=large]
\begin{array}{rl} \min\limits_{x} & c^\top \cdot x \\ \text{s.t.} & A \cdot x \geq b \end{array} \arrow[r, phantom, "\Longleftrightarrow"] \arrow[d, leftrightarrow, "\text{dualize}"] & \begin{array}{rl} \min\limits_{x^+, x^-} & [c^\top, -c^\top] \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \\ \text{s.t.} & [A, -A] \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \geq b \\ & x^+, x^- \geq 0 \end{array} \arrow[d, leftrightarrow, "\text{dualize}"] \\
\begin{array}{rl} \max\limits_{p} & p^\top b \\ \text{s.t.} & p^\top A = c^\top \\ & p \geq 0 \end{array} \arrow[r, phantom, "\Longleftrightarrow"] & \begin{array}{rl} \max\limits_{p} & p^\top b \\ \text{s.t.} & p^\top [A, -A] \leq [c^\top, -c^\top] \\ & p \geq 0 \end{array}
\end{tikzcd}$$

Proof of Theorem 7.2 (ii)

Proof: ii introducing a slack variable

$$\begin{tikzcd}
\begin{array}{ll} \min\limits_{x} & c^\top \cdot x \\ \text{s.t.} & A \cdot x \geq b \end{array} \arrow[r, Leftrightarrow] \arrow[d, leftrightarrow, "\text{dualize}"] & \begin{array}{ll} \min\limits_{x,y} & c^\top x + \mathbf{0}^\top y \\ \text{s.t.} & Ax - Iy = b \\ & y \geq 0 \end{array} \arrow[d, leftrightarrow, "\text{dualize}"] \\
\begin{array}{ll} \max\limits_{p} & b^\top \cdot p \\ \text{s.t.} & A^\top \cdot p = c \\ & p \geq 0 \end{array} \arrow[r, Leftrightarrow] & \begin{array}{ll} \max\limits_{p} & p^\top b \\ \text{s.t.} & A^\top \cdot p = c \\ & (-I)^\top \cdot p \leq \mathbf{0} \end{array}
\end{tikzcd}$$

Proof of Theorem 7.2 (iii)

Proof: iii eliminating linear combination row

$$\begin{aligned} \min_{x} \quad & c^\top \cdot x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned}$$

with $a_m^\top = \sum_{i=1}^{m-1} \gamma_i a_i^\top$,
$b_m = \sum_{i=1}^{m-1} \gamma_i b_i$

$$\iff$$

$$\begin{aligned} \min_{x} \quad & c^\top x \\ \text{s.t.} \quad & \begin{pmatrix} a_1^\top \\ \vdots \\ a_{m-1}^\top \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_{m-1} \end{pmatrix} \\ & x \geq 0 \end{aligned}$$

$\updownarrow$dualize

$\updownarrow$dualize

$$\begin{aligned} \max_{p} \quad & \sum_{i=1}^{m} p_i b_i = \sum_{i=1}^{m-1} (p_i + \gamma_i p_m) b_i \\ \text{s.t.} \quad & \sum_{i=1}^{m} p_i a_i^\top = \sum_{i=1}^{m-1} (p_i + \gamma_i p_m) a_i^\top \leq c^\top \end{aligned}$$

$$\iff$$

$$\begin{aligned} \max_{q} \quad & \sum_{i=1}^{m-1} q_i b_i \\ \text{s.t.} \quad & \sum_{i=1}^{m-1} q_i a_i^\top \leq c^\top \end{aligned}$$

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7.3 Weak and Strong Duality

Weak Duality Theorem

Theorem 7.3 If x is a feasible solution to the primal LP (minimization problem) and p a feasible solution to the dual LP (maximization problem), then

$$c^\top \cdot x \geq p^\top \cdot b.$$

Proof:

- here only for special case

$$\begin{array}{rl} \min\limits_{x} & c^\top \cdot x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array} \quad \xleftrightarrow{\text{dualize}} \quad \begin{array}{rl} \max\limits_{p} & p^\top b \\ \text{s.t.} & p^\top A \leq c^\top \\ & p \geq 0 \end{array}$$

- if x is feasible for primal LP and p feasible for dual LP, then

$$c^\top x \geq (p^\top A)x = p^\top (Ax) \geq p^\top b$$

$\square$

A Corollary

Corollary 7.4 Consider a primal-dual pair of linear programs as above.

- **a** If the primal LP is unbounded (i.e., optimal cost = $-\infty$), then the dual LP is infeasible.
- **b** If the dual LP is unbounded (i.e., optimal cost = ∞), then the primal LP is infeasible.
- **c** If x and p are feasible solutions to the primal and dual LP, resp., and if $c^\top \cdot x = p^\top \cdot b$, then x and p are optimal solutions.

Strong Duality Theorem

Theorem 7.5 If an LP has an optimal solution, so does its dual and the optimal costs are equal.

Proof:

- here only for an LP in standard form

$$\begin{aligned} \min_x \quad & c^\top \cdot x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \quad \overset{\text{dualize}}{\longleftrightarrow} \quad \begin{aligned} \max_p \quad & p^\top b \\ \text{s.t.} \quad & p^\top A \leq c^\top \end{aligned}$$

- w.l.o.g., assume that rows of A are linearly independent
- Simplex algorithm finds optimal basis B, i.e.,

$$\bar{c}^\top = c^\top - c_B A_B^{-1} A \geq 0 \tag{7.1}$$

  and corresponding optimal basic feasible solution $x = (x_B, x_N)$ with $x_B = A_B^{-1} b$, $x_N = 0$

- for $p^\top := c_B^\top A_B^{-1}$, we have $c^\top \overset{(7.1)}{\geq} p^\top A$, i.e. p is a feasible dual solution with cost $p^\top b = c_B^\top A_B^{-1} b = c_B^\top x_B = c^\top x$ $\square$

Possible Outcomes of Primal and Dual LPs

Different Possibilities for Primal and Dual LP

primal \ dual	finite optimum	unbounded	infeasible
finite optimum	possible	impossible	impossible
unbounded	impossible	impossible	possible
infeasible	impossible	possible	possible

Corollary Let $p^* = \inf_x\{c^\top x \,:\, Ax \geq b\} \in \mathbb{R} \cup \{-\infty, \infty\}$ and $d^* = \sup_p\{b^\top p \,:\, A^\top p = c,\ p \geq 0\} \in \mathbb{R} \cup \{-\infty, \infty\}$. Then, it either holds

$$(p^* = d^*) \qquad \text{or} \qquad (p^* = \infty, d^* = -\infty).$$

Example of infeasible primal and dual LP:

$$\begin{aligned} \min_x \quad & x_1 + 2\,x_2 \\ \text{s.t.} \quad & x_1 + \, x_2 = 1 \\ & 2\,x_1 + 2\,x_2 = 3 \end{aligned} \qquad\qquad \begin{aligned} \max_p \quad & p_1 + 3\,p_2 \\ \text{s.t.} \quad & p_1 + 2\,p_2 = 1 \\ & p_1 + 2\,p_2 = 2 \end{aligned}$$

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7.4 Complementary Slackness

Complementary Slackness

Consider the following pair of primal and dual LPs:

$$\begin{aligned} \min_x \quad & c^\top \cdot x \\ \text{s.t.} \quad & A \cdot x \geq b \end{aligned} \qquad\qquad \begin{aligned} \max_p \quad & p^\top \cdot b \\ \text{s.t.} \quad & p^\top \cdot A = c^\top \\ & p \geq 0 \end{aligned}$$

If x and p are feasible solutions, then $c^\top \cdot x = p^\top \cdot A \cdot x \geq p^\top \cdot b$.Thus,

$$c^\top \cdot x = p^\top \cdot b \quad \Longleftrightarrow \quad \text{for all } i\text{: } p_i = 0 \text{ if } a_i^\top \cdot x > b_i.$$

Theorem 7.6 Consider an arbitrary pair of primal and dual LPs. Let x and p be feasible solutions to the primal and dual LP, respectively. Then x and p are both optimal if and only if

$$\begin{aligned} u_i &:= p_i\,(a_i^\top \cdot x - b_i) = 0 \quad \text{for all } i, \\ v_j &:= (c_j - p^\top \cdot A_j)\,x_j = 0 \quad \text{for all } j. \end{aligned}$$

Proof of Theorem 7.6

Proof:

$$\begin{aligned} u_i &:= p_i(a_i^\top x - b_i) \\ v_j &:= (c_j - p^\top A_j)x_j \end{aligned}$$

- for any primal-dual pair, we have $u_i \geq 0$ for all i and $v_j \geq 0$ for all j
- we have

$$\begin{aligned} \sum_i u_i &= \sum_i p_i(a_i^\top x - b_i) = p^\top A x - p^\top b \\ \sum_j v_j &= \sum_j (c_j - p^\top A_j)x_j = c^\top x - p^\top A x, \end{aligned}$$

  in particular $c^\top x - p^\top b = \sum_i u_i + \sum_j v_j$
- by strong duality, if x and p are optimal, then $c^\top x - p^\top b = 0$ and, hence $u_i = 0$ for all i and $v_j = 0$ for all j
- on the other hand, if $u_i = 0$ for all i and $v_j = 0$ for all j, then $c^\top x = p^\top b$ and x and p are optimal by weak duality

□

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7.5 Geometry of Duality

Geometric View on Duality

Consider pair of primal and dual LPs with $A \in \mathbb{R}^{m \times n}$ and $\operatorname{rank}(A) = n$:

$$\begin{aligned} \min_x \quad & c^\top \cdot x \\ \text{s.t.} \quad & A \cdot x \geq b \end{aligned} \qquad\qquad \begin{aligned} \max_p \quad & p^\top \cdot b \\ \text{s.t.} \quad & p^\top \cdot A = c^\top \\ & p \geq 0 \end{aligned}$$

Let $I \subseteq \{1, \ldots, m\}$ with $|I| = n$ and a_i, $i \in I$, linearly independent.

$\Longrightarrow \quad a_i^\top \cdot x = b_i$, $i \in I$, has unique solution x^I (basic solution)

Assume that x^I is nondegenerate, i.e., $a_i^\top \cdot x \neq b_i$ for $i \notin I$.

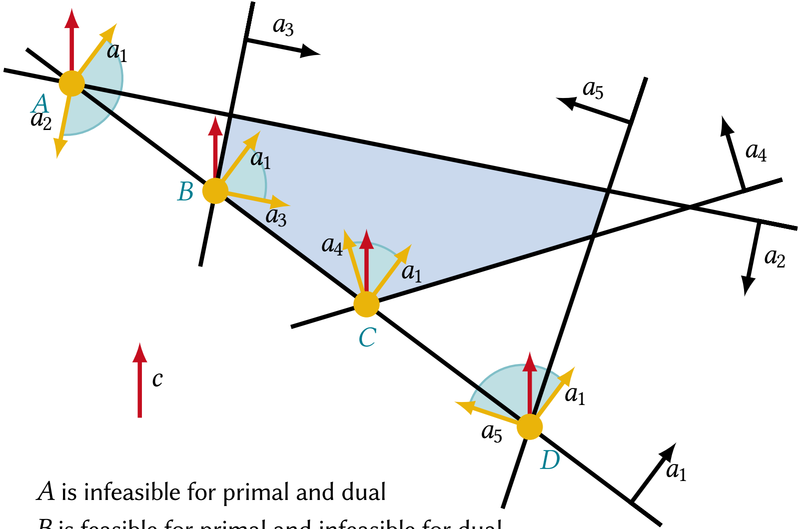
Let $p \in \mathbb{R}^m$ (dual vector). Then x, p are optimal solutions if

- i $a_i^\top \cdot x \geq b_i$ for all i (primal feasibility)
- ii $p_i = 0$ for all $i \notin I$ (complementary slackness)
- iii $\sum_{i=1}^m p_i \cdot a_i = c$ (dual feasibility)
- iv $p \geq 0$ (dual feasibility)

note that ii and iii imply $\sum_{i \in I} p_i \cdot a_i = c$ which has a unique solution p^I.

The a_i, $i \in I$, form basis for dual LP and p^I is corresponding basic solution.

Geometric View on Duality (Cont.)



A is infeasible for primal and dual

B is feasible for primal and infeasible for dual

C is feasible for primal and dual

D is infeasible for primal and feasible for dual

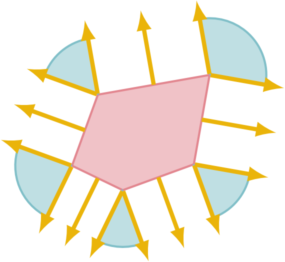
Outer Normal Cone

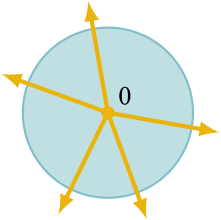
Definition 7.7 Let x^* be a boundary point of the polyhedron $P = \{x \mid A \cdot x \leq b\}$ and let I denote the (non-empty) set of indices of rows $a_i^\top$ of A with $a_i^\top x^* = b_i$. The conic hull

$$\text{cone}\{a_i \mid i \in I\} := \left\{ \sum\nolimits_{i \in I} \lambda_i a_i \mid \lambda_i \geq 0 \ \forall i \in I \right\}$$

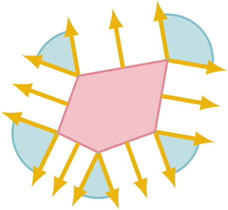
is the outer normal cone of P in x^*.

Example.





Properties of Outer Normal Cones



Observation 7.8 Consider a polyhedron $P \subseteq \mathbb{R}^n$.

- a For some $c \in \mathbb{R}^n \setminus \{0\}$ a point $x \in P$ maximizes $c^\top x$ over P if and only if c is in the outer normal cone of P in x.
- b If x is a boundary point of P, the polyhedral cone consisting of the feasible directions at x is polar to the outer normal cone of P in x; that is, inner products are non-positive.
- c If $P \subseteq \mathbb{R}^n$ is a polytope, every point in $\mathbb{R}^n$ is in the outer normal cone of some vertex. The interiors of these cones do not intersect.

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7.6 Marginal Costs

Dual Variables as Marginal Costs

Consider the primal dual pair:

$$\begin{aligned} \min_x \quad & c^\top \cdot x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \max_p \quad & p^\top \cdot b \\ \text{s.t.} \quad & p^\top \cdot A \leq c^\top \end{aligned}$$

Let x^* be optimal basic feasible solution to primal LP with basis B, i.e., $x_B^* = A_B^{-1} \cdot b$ and assume that $x_B^* > 0$ (i.e., x^* non-degenerate).

Replace b by $b + d$. For small d, the basis B remains feasible and optimal:

$$A_B^{-1} \cdot (b + d) = A_B^{-1} \cdot b + A_B^{-1} \cdot d \geq 0 \qquad \text{(feasibility)}$$

$$\bar{c}^\top = c^\top - c_B^\top \cdot A_B^{-1} \cdot A \geq 0 \qquad \text{(optimality)}$$

Optimal cost of perturbed problem is

$$c_B{}^\top \cdot A_B^{-1} \cdot (b + d) = c_B{}^\top \cdot x_B^* + \underbrace{(c_B^\top \cdot A_B^{-1})}_{=p^\top} \cdot d$$

Thus, p_i is the marginal cost per unit increase of b_i.

Dual Variables as Shadow Prices

Diet problem:

- a_{ij} := amount of nutrient i in one unit of food j
- b_i := requirement of nutrient i in some ideal diet
- c_j := cost of one unit of food j on the food market

LP duality: Let x_j := number of units of food j in the diet:

$$\begin{aligned} \min_x \quad & c^\top \cdot x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \max_p \quad & p^\top \cdot b \\ \text{s.t.} \quad & p^\top \cdot A \leq c^\top \end{aligned}$$

Dual interpretation:

- p_i is "fair" price per unit of nutrient i
- $p^\top \cdot A_j$ is value of one unit of food j on the nutrient market
- food j used in ideal diet $(x_j^* > 0)$ is consistently priced at the two markets (by complementary slackness)
- ideal diet has the same cost on both markets (by strong duality)

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7.7 Dual Simplex

Dual Basic Solutions

Consider LP in standard form with $A \in \mathbb{R}^{m \times n}$, $\operatorname{rank}(A) = m$, and dual LP:

$$\begin{aligned} \min_x \quad & c^\top \cdot x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \max_p \quad & p^\top \cdot b \\ \text{s.t.} \quad & p^\top \cdot A \leq c^\top \end{aligned}$$

Observation 7.9 A basis B yields

- a primal basic solution given by $x_B := A_B^{-1} \cdot b$ and
- a dual basic solution $p^\top := c_B^\top \cdot A_B^{-1}$.

Moreover,

a. the values of the primal and the dual basic solutions are equal:

$$c_B^\top \cdot x_B = c_B^\top \cdot A_B^{-1} \cdot b = p^\top \cdot b;$$

b. p is feasible if and only if $\bar{c} = c - p^\top A \geq 0$;

c. reduced cost $\bar{c}_i = 0$ corresponds to active dual constraint;

d. p is degenerate if and only if $\bar{c}_i = 0$ for some non-basic variable x_i.

Dual Simplex Method

- Let B be a basis whose corresponding dual basic solution p is feasible.
- If also the primal basic solution x is feasible, then x, p are optimal.
- Assume that $x_{B(\ell)} < 0$ and consider the ℓth row of the simplex tableau

$$(x_{B(\ell)}, v_1, \dots, v_n) \qquad \text{(pivot row)}$$

I Let $j \in \{1, \dots, n\}$ with $v_j < 0$ and

$$\frac{\bar{c}_j}{|v_j|} = \min_{i\,:\,v_i<0} \frac{\bar{c}_i}{|v_i|}$$

Performing an iteration of the simplex method with pivot element v_j yields new basis B' and corresponding dual basic solution p' with

$$c_{B'}^{\top} \cdot A_{B'}^{-1} \cdot A \leq c^{\top} \quad \text{and} \quad p'^{\top} \cdot b \geq p^{\top} \cdot b \quad (\text{with} > \text{if } \bar{c}_j > 0).$$

II If $v_i \geq 0$ for all $i \in \{1, \dots, n\}$, then the dual LP is unbounded and the primal LP is infeasible.

Remarks on the Dual Simplex Method

- Dual simplex method terminates if lexicographic pivoting rule is used:
  - Choose any row ℓ with $x_{B(\ell)} < 0$ to be the pivot row.
  - Among all columns j with $v_j < 0$ choose the one which is lexicographically minimal when divided by $|v_j|$.

- Dual simplex method is useful if, e.g., dual basic solution is readily available.

- Example: Resolve LP after right-hand-side b has changed.