Approximation Algorithms (ADM III)
1- Introduction

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Outline

Organization

- Short Videos on ISIS

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- Weekly Zoom-Slot
  - Wednesday 12:00 (noon) ?
  - FAQ-sessions
  - Addional material
  - Exercises

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- Set of problems as homework once in a while
- Final oral exam in spring during the semester break (details t.b.a.)

Literature

The course is based on selected parts of the textbook:

- David P. Williamson, David B. Shmoys, The Design of Approximation Algorithms, Cambridge University Press, 2011

Further reading:

- V. V. Vazirani, Approximation Algorithms, Springer Verlag, 2001
- G. Ausiello, P. Crescenzi, G. Gambosi, V. Kann, A. Marchetti-Spaccamela, M. Protasi, Complexity and Approximation: Combinatorial Optimization Problems and Their Approximability Properties, Springer Verlag, 1999
- D. S. Hochbaum (ed.), Approximation Algorithms for NP-Hard Problems, PWS Publishing Company, 1995
- B. Korte, J. Vygen, Combinatorial Optimization, Springer Verlag, fifth edition, 2012

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What is "good enough"?
We would like to have some sort of a priori performance guarantee.

Approximation Algorithms

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- $\alpha > 1$ for minimization problems, and
- $\alpha < 1$ for maximization problems.

Thus, a $\frac{1}{2}$-approximation algorithm for a maximization problem is a polynomial-time algorithm that always returns a solution whose value is at least half the value of the optimum value.

Approximation Algorithms

Given an optimization problem, we'll often use the notation $\texttt{ALG}$ to denote the value returned by a given algorithm for this problem, and $\texttt{OPT}$ to denote the optimum value of the problem.

Example 1.2

For a maximization problem, an α-approximation algorithm satisfies

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Example:

Constructing an inclusion-wise maximal matching is a $\frac{1}{2}$-approx. algorithm for the maximum cardinality matching problem.

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- Often, approximation algorithms are much better in practice than indicated by their performance guarantee.
- Nevertheless, the study of worst-case bounds provides a mathematically rigourous basis to analyse heuristics, and to undersand *what can go wrong*.
- Approximation factors and inapproximability results give a kind of *metric* to compare the hardness of discrete optimization problems.

Overview of lecture

Learn *techniques* to design approximation algorithms

- Greedy algorithms
- Dynamic Programming & Data rounding
- Linear Programming based techniques (dual fitting, primal-dual algos)
- Deterministic & Randomized Rounding
- Reductions to show *hardness of approximation*

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Approximation results for many standard discrete problems, e.g.

- Set cover
- Scheduling
- Facility location
- Traveling Salesperson Problem
- Clustering
- ...

Polynomial-Time Approximation Schemes

Definition 1.3 (PTAS)

A polynomial-time approximation scheme (PTAS) is a family of algorithms $\{A_\varepsilon\}$, where there is an algorithm for each $\varepsilon > 0$, such that A_ε is a

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Remark: The running time of A_ε may depend badly on ε, e.g. $\{A_\varepsilon\}$ is a PTAS if A_ε runs in $O(n^{1/\varepsilon})$ or even $O(n^{\exp(1/\varepsilon)})$.

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Definition 1.4 (FPTAS)

A fully polynomial-time approximation scheme (FPTAS) is a family of algorithms $\{A_\varepsilon\}$, where there is an algorithm for each $\varepsilon > 0$, such that the running time of A_ε is polynomial in both the problem size n and $\frac{1}{\varepsilon}$, and A_ε is a $(1+\varepsilon)$-approximation algorithm for minimization problems, (resp. a $(1-\varepsilon)$-approximation algorithm for maximization problems).

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Examples: FPTASes exist for the Knapsack Problem and for scheduling on a fixed number of identical machines.

Remarks: The running time of an FPTAS is of the form $O(1/\varepsilon^k \cdot n^d)$, so in some sense, the precision ε only influences the hidden constant of the polynomial.

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Set Cover Problem

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Set Cover Problem:
Given: A set of elements $E = \{e_1, \dots, e_n\}$, a family of subsets $\{S_1, \dots, S_m\} \subseteq 2^E$, and a weight $w_j \geq 0$ for each $j \in \{1, \dots, m\}$.

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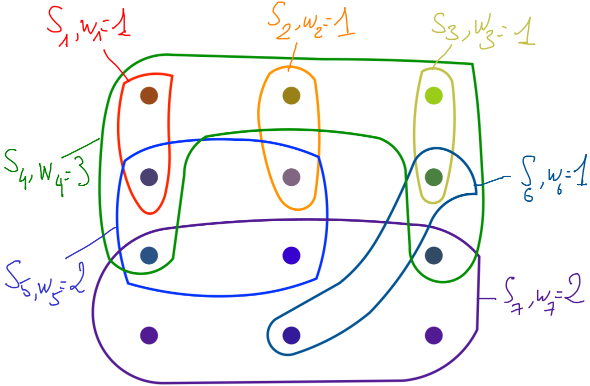
If $w_j = 1$ for each $j \in \{1, \ldots, m\}$, the problem is called Unweighted Set Cover Problem.

Set Cover



Cost of two set covers:

- S_1, S_2, S_3, S_7: $W = 5$ (minimal cost for this example)
- S_2, S_4, S_7: $W = 6$

Outline

IP Formulation of Set Cover

Formulation as integer linear program:

$$
\begin{aligned}
z_{IP}^* = \quad \min \quad & \sum_{j=1}^{m} w_j \cdot x_j \\
\text{s.t.} \quad & \sum_{j:e_i \in S_j} x_j \geq 1 \quad \text{for all } i = 1, \ldots, n \\
& x_j \in \{0, 1\} \quad \text{for all } j = 1, \ldots, m
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Linear programming relaxation:

$$z_{LP}^* = \min \sum_{j=1}^{m} w_j \cdot x_j$$

$$\text{s.t.} \sum_{j: e_i \in S_j} x_j \geq 1 \quad \text{for all } i = 1, \ldots, n \tag{2}$$

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Theorem 1.6

The rounding algorithm above is an f-approximation algorithm for the Set Cover Problem.

Proof...

Vertex cover

Input: Graph $G = (V, E)$, weight $w_i \geq 0$ for all $i \in V$.

Task: Find $I \subset V$ minimizing $w(I) := \sum_{i \in I} w_i$,

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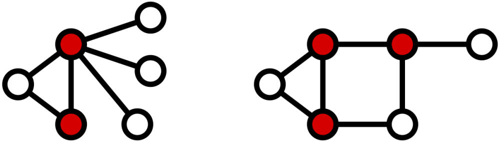

Some *vertex covers* (minimal for unweighted problem, i.e., $w_i = 1$)

Exercise 1.7

Use the previous algorithm to design an approximation algorithm for VERTEX COVER. What is its performance guarantee ?

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Dual linear program:

$$\begin{aligned} z^*_{LP} = \quad \max \quad & \sum_{i=1}^{n} y_i \\ \text{s.t.} \quad & \sum_{e_i \in S_j} y_i \leq w_j \qquad \text{for all } j = 1, \ldots, m \\ & y_i \geq 0 \qquad \text{for all } i = 1, \ldots, n \end{aligned} \tag{3}$$

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Remark: Notice that $I^* \supseteq \hat{I}$ (the solution obtained by rounding the primal LP) due to complementary slackness!

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Primal-dual algorithm for the Set Cover Problem:

1. set $y :\equiv 0$ and $I := \emptyset$;
2. while $\exists e_k \notin \bigcup_{j \in I} S_j$
3. increase y_k until $\exists j$ with $e_k \in S_j$ such that $\sum_{i : e_i \in S_j} y_i = w_j$;
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4. $\quad$ set $I := I \cup \{\ell\}$;
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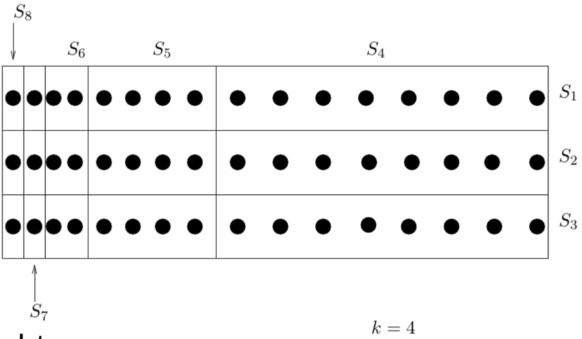
Theorem 1.11

The greedy algorithm returns a cover I with $w(I) \leq H_g \cdot z^*_{LP}$, where

$$g := \max_j |S_j| \text{ and } H_g := \sum_{k=1}^{g} \frac{1}{k} \approx \ln\ g.$$

Tightness of approximation factor



Instance:

- unit weights
- $E = \{(i,j) : i \in [3],\ j \in [2^k]\}$
- $S_1 = \{(1,j) j \in [2^k]\}, S_2 = \{(2,j) j \in [2^k]\}, S_3 = \{(3,j) j \in [2^k]\}$
- S_{3+j} contains the points of 2^{k-j} "columns", $\forall j \in [k]$
- S_{k+4} contains the points of the last remaining column.

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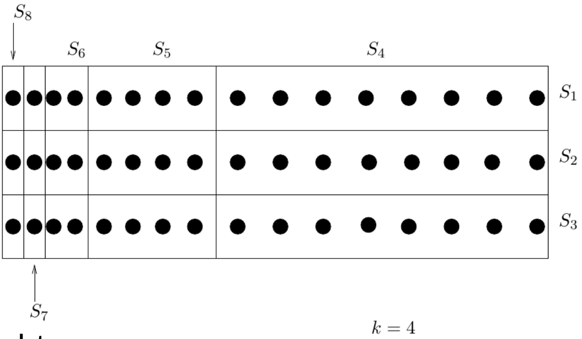

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GREEDY selects $S_4, \ldots, S_{4+k}$ but the optimal set cover is (S_1, S_2, S_3), hence an approx. ratio of $(k+1)/3 = O(\log(n))$.

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Proof...

Outline

Hardness of Approximability Results for Set Cover

Theorem 1.14(Lund & Yannakakis 1994)

If there is a $(c \ln n)$-approximation algorithm for the Unweighted Set Cover Problem for some constant $c < 1$, then there is an $O(n^{O(\log \log n)})$-time deterministic algorithm for each NP-complete problem.

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Theorem 1.15 (Feige 1998)

There is some constant $c > 0$ such that if there is a $(c \ln n)$-approximation algorithm for the Unweighted Set Cover Problem, then $P = NP$.

Hardness of Approximability Results for Vertex Cover

Theorem 1.16 (Dinur & Safra 2002)

If there is an α-approximation algorithm for the Vertex Cover Problem with $\alpha < 10\sqrt{5} - 21 \approx 1.36$, then $P = NP$.

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Theorem 1.17 (Khot & Regev 2008)

Assuming the Unique Games Conjecture holds, if there is an α-approximation algorithm for the Vertex Cover Problem with $\alpha < 2$, then $P = NP$.