Approximation Algorithms (ADM III)

2- Local Search & Greedy Algorithms

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Outline

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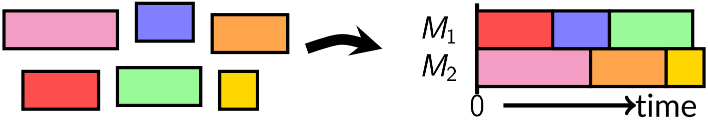
- Schedules can be represented by Gantt charts



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  - Other environments: Q: uniform parallel machines, F: flow shop, J: Job shop,...

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  - Other obvious specifications (e.g., $p_j = 1$ means that all jobs have unit processing times)

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    - Many other criterions (flow time $F_j = C_j - r_j$, tardiness $T_j = \max(0, C_j - d_j)$ of job j, unit penalty for tardy jobs $U_j = 1_{C_j > d_j}$, ...)

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Example 2.1

- $1|r_j|C_{\max}$: minimize the makespan on a single machine; jobs have release dates.
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- $P||\sum w_j Cj$: minimize the weighted sum of completion times on parallel identical machines.
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- $P3|\mathtt{prec}, d_j = d|\sum w_j Lj$: minimize the total weighted lateness on 3 parallel identical machines. Jobs must respect precedences and have a common deadline $d_j = d$.
  Ex. input: $n = 6$, $p = (1, 2, 1, 5, 4, 3)$, $d = 2$, $\{1 \prec 2 \prec 4, 1 \prec 3\}$.

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1 Introduction to Scheduling Problems

2 Scheduling Jobs with Due Dates on a Single Machine

3 The *k*-Center Problem

4 Scheduling Jobs on Identical Parallel Machines

5 The Traveling Salesperson Problem (TSP)

6 Greedy Maximization of Submodular Functions

7 Minimum Edge Coloring

Scheduling Jobs with Due Dates on a Single Machine

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- In other words, we consider the problem

$$1|r_j|L_{\max}.$$

Scheduling Jobs with Due Dates on a Single Machine

- Minimize the maximum lateness on one machine with release dates: $1|r_j|L_{\max}$.

Theorem 2.2

Deciding whether $L^*_{\max} \leq 0$ is strongly *NP*-complete.

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Greedy 2-Approximation Algorithm for Negative Due Dates

For a subset of jobs $S \subseteq \{1, \dots, n\}$ let:

$$r(S) := \min_{j \in S} r_j \qquad p(S) := \sum_{j \in S} p_j \qquad d(S) := \max_{j \in S} d_j$$

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Let $L^*_{\max}$ denote the optimal value. For each subset S of jobs

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The k-Center Problem

Given: A finite metric space V with distances d_{ij} for $i, j \in V$ and $k \in \mathbb{N}$.

Task: Find k centers in V, i.e., $S \subseteq V$ with $|S| = k$.

Objective: Minimize $\max_{i \in V} d(i, S)$ where $d(i, S) := \min_{j \in S} d_{ij}$.

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Theorem 2.6

The algorithm is a 2-approximation algorithm for the k-Center Problem.

The k-Center Problem: hardness of approximation

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Proof: Reduction from Dominating Set Problem…

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Given: n jobs $j = 1, \dots, n$ with processing time $p_j \geq 0$, $j = 1, \dots, n$, and m identical parallel machines.

Task: Process each job j nonpreemptively for p_j units of time on one of the m machines. A machine can process at most one job at a time.

Objective: Minimize the maximum machine load, i.e., the maximum completion time $C_{\max} := \max\limits_{j=1,\dots,n} C_j$ (makespan).

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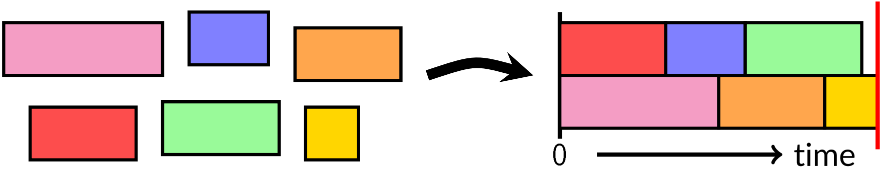

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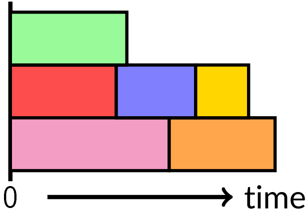

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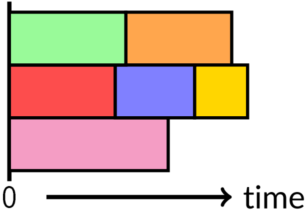

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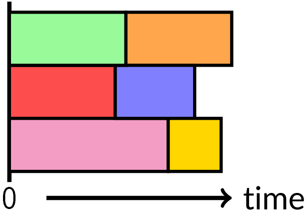

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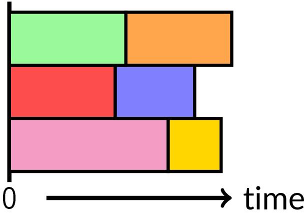

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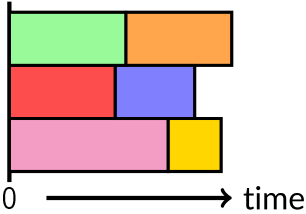

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Lemma 2.10

If the Local Search Algorithm always moves job j to a currently least loaded machine, it terminates after at most n iterations.

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Later: There is a PTAS for this scheduling problem!

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Objective: Minimize total length of tour: $d_{\pi(n)\pi(1)} + \sum_{i=1}^{n-1} d_{\pi(i)\pi(i+1)}$

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In the following we thus consider the metric TSP where distances between cities fulfill the triangle inequalities.

Lower Bounds for TSP

Lemma 2.14

The cost of a minimum spanning tree of the complete graph with nodes V and edge costs d_{ij} is a lower bound on the length of a shortest TSP tour.

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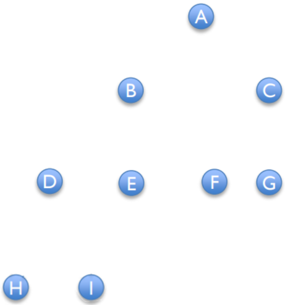

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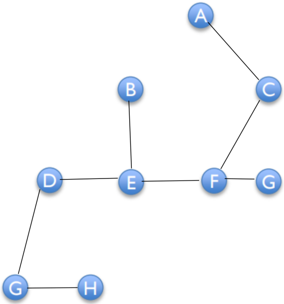

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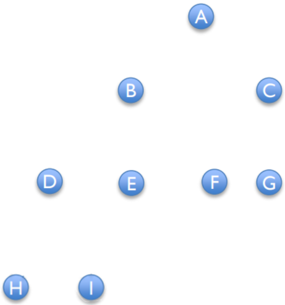

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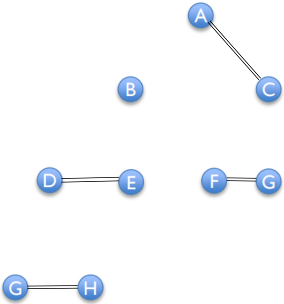

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Let $S \subseteq V$ with $|S|$ even and consider the complete graph on nodes S and edge costs d_{ij}. Twice the cost of a min-cost perfect matching of S is a lower bound on the length of a shortest TSP tour of V.

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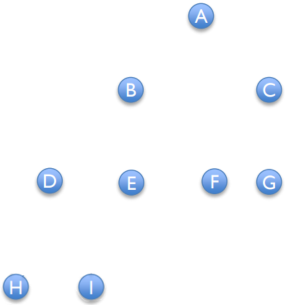

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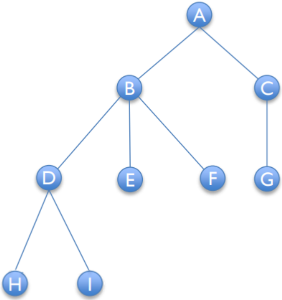

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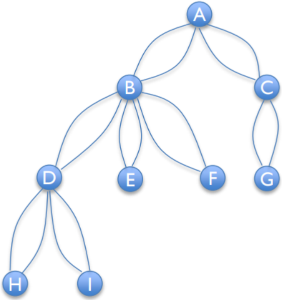

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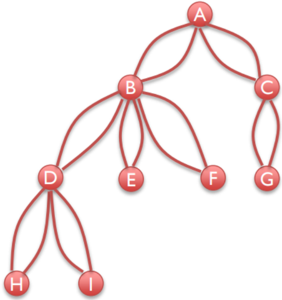

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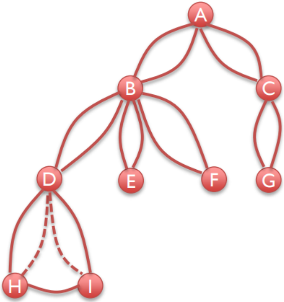

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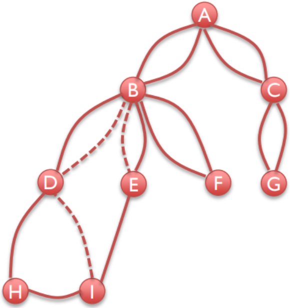

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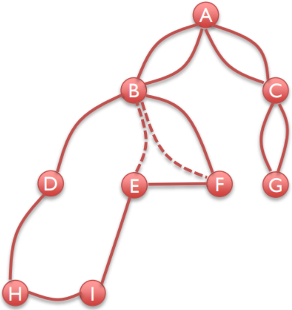

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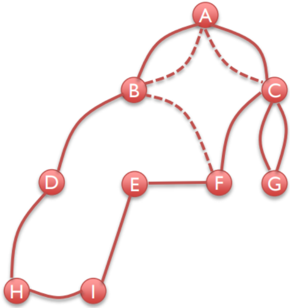

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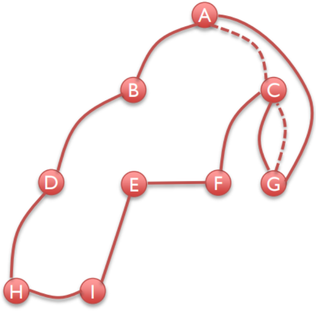

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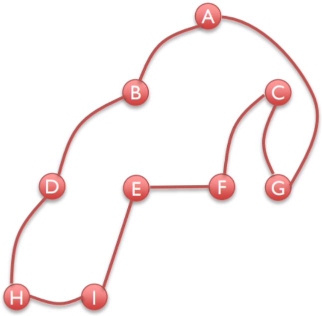

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Theorem 2.16

The Double-Tree Algorithm is a 2-approximation algorithm for the TSP.

TSP: Nearest Insertion Algorithm

Nearest Insertion Algorithm

1. start with tour through two cities $S := \{i, j\}$ of minimum distance d_{ij};
2. for each uninserted city k, compute the minimum distance $d(k, S)$ between k and a city in the current tour;
3. let $\ell := \arg\min_{k \notin S} d(k, S)$; add ℓ to the tour after its nearest city;
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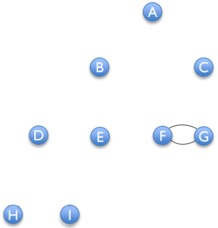

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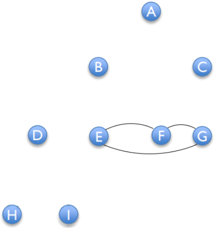

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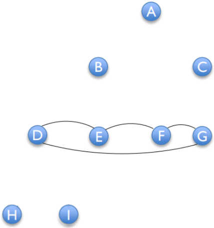

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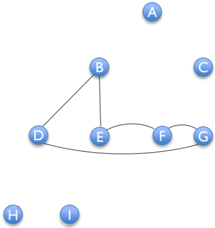

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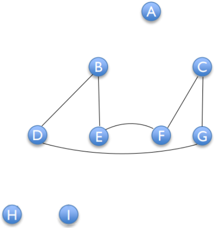

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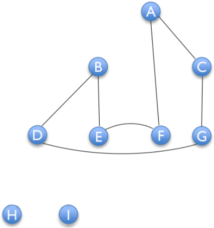

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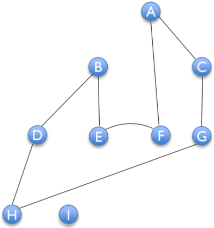

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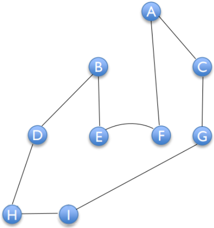

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The Nearest Insertion Algorithm is a 2-approximation algorithm for the metric TSP.

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Notice: The Nearest Insertion Algorithm is closely related to Prim's Algorithm and the Double-Tree Algorithm.

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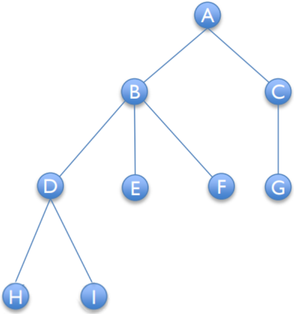

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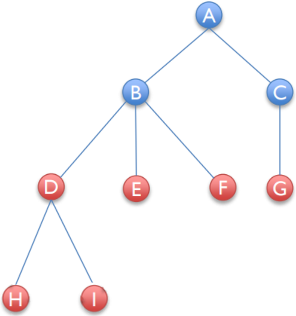

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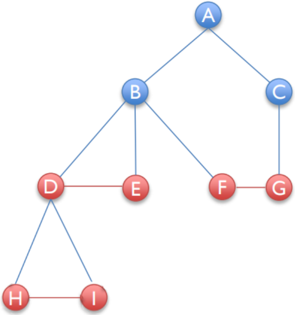

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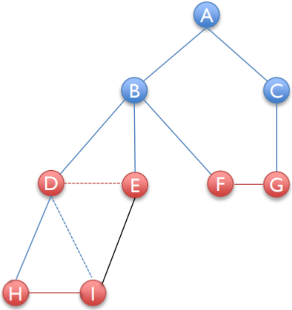

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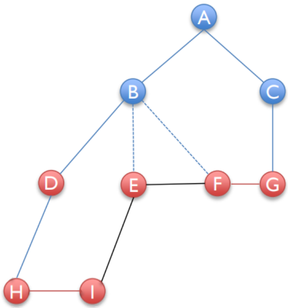

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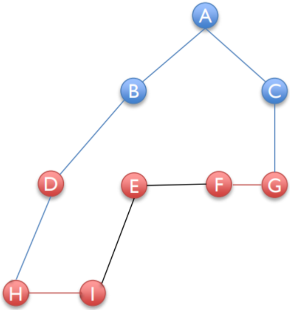

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Theorem 2.18

Christofides' Algorithm is a $\frac{3}{2}$-approximation algorithm for the TSP.

Proof:… $\square$

Approximability of Metric TSP

Theorem 2.19 (Papadimitriou & Vempala 2006)

There is no α-approximation algorithm for the metric TSP for $\alpha < 220/219$, unless $P = NP$.

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Theorem 2.20 (Karlin, Klein, Gharan 2020)

A $(3/2 - \varepsilon)$-approximation algorithm for some $\varepsilon > 10^{-36}$.

Approximability of Metric TSP

Theorem 2.19 (Papadimitriou & Vempala 2006)

There is no α-approximation algorithm for the metric TSP for $\alpha < 220/219$, unless $P = NP$.

Theorem 2.20 (Karlin, Klein, Gharan 2020)

A $(3/2 - \varepsilon)$-approximation algorithm for some $\varepsilon > 10^{-36}$.

Remark: There is a PTAS for the Euclidean TSP (special case).

Outline

1. Introduction to Scheduling Problems
2. Scheduling Jobs with Due Dates on a Single Machine
3. The k-Center Problem
4. Scheduling Jobs on Identical Parallel Machines
5. The Traveling Salesperson Problem (TSP)
6. **Greedy Maximization of Submodular Functions**
7. Minimum Edge Coloring

Submodular Function

Given a finite ground set V, we consider set functions $f : 2^V \to \mathbb{R}$, i.e. we assign each $S \subseteq V$ a value $f(S) \in \mathbb{R}$.
We assume that $f(S)$ can be evaluated in constant time, for each $S \subseteq V$.

Definition 2.21

A function $f : 2^V \to \mathbb{R}$ is submodular if for every $A \subseteq B \subseteq V$ and $i \in V \setminus B$ it holds

$$f(A \cup \{i\}) - f(A) \geq f(B \cup \{i\}) - f(B).$$

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Equivalently, it can be seen that f is submodular iff

$$f(A \cup B) + f(A \cap B) \leq f(A) + f(B), \quad \forall A, B \subseteq V.$$

In words, f is submodular if it exhibits a *diminishing returns* property.

Submodular Function Maximization

We consider the problem of maximizing a nonnegative monotone submodular function, i.e., a submodular function f such that

$$0 \leq f(A) \leq f(B), \qquad \forall A \subseteq B \subseteq V,$$

subject to a cardinality constraint:

$$\max_{S \subseteq V} \{ f(S) : \ |S| \leq k \}.$$

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This problem generalizes many classical problems in discrete mathematics !

Submodular Function Maximization

$$\max_{S\subseteq V}\{f(S):\ |S|\leq k\},\qquad f \text{ nonnegative monotone submodular.}$$

Special cases

- Linear (aka modular) functions: $f(S)=\sum_{i\in S} w_i$ for some $w_i \geq 0$

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- The rank function of a matroid

$$f(S)=\max\{|U|:\ U\subseteq S,\ U \text{ independent}\}.$$

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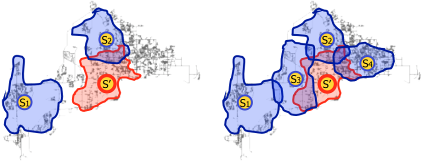
Applications

- Sensor location
- Antenna selection



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- The rank function of a matroid
- Weighted coverage functions
- Facility location:

  There is a collection of n potential locations to open facilities to serve m customers. Opening a facility at location j provides service of value $M_{i,j}\geq 0$ to customer i. If we open the subset of facilities $S\subseteq\{1,\ldots,n\}$ and each customer selects the opened facility with highest value, the total value provided to all customers is

  $$f(S)=\sum_{i\in[m]}\max_{j\in S} M_{ij}$$

Greedy Algorithm for Submodular Optimization

$$\max_{S\subseteq V}\{f(S):\ |S|\leq k\},\qquad f\text{ nonnegative monotone submodular.}$$

Greedy Algorithm

```
$S_0 \leftarrow \emptyset$
For $i = 1, \dots k$:
    $e_i \leftarrow \operatorname{argmax}_{e\in V}\ f(S_{i-1}\cup\{e\}) - f(S_{i-1})$
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Proof

Submodular function maximization: further results

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A modified greedy algorithm achieves an approximation ratio of $(1-1/e)$ maximizing monotone submodular functions subject to a *knapsack constraint* $\sum_{i \in S} w_i \leq B$.

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Theorem 2.26 (Buchbinder, Feldman, Naor, Schwartz 2014)

A $1/e-$approx. algo for maximizing non-monotone ≥ 0 submodular funct.

Outline

Minimum Edge Coloring

Given: Graph $G = (V, E)$ without parallel edges

Task: Find a coloring $c : E \to \{1, \dots, C\}$ such that no two incident edges get the same color.

Objective: Use a minimum number of colors $C = \chi'(G)$.

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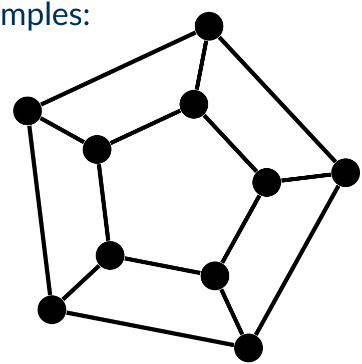

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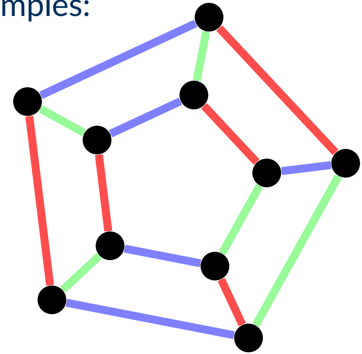

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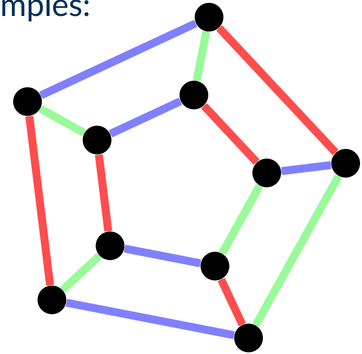

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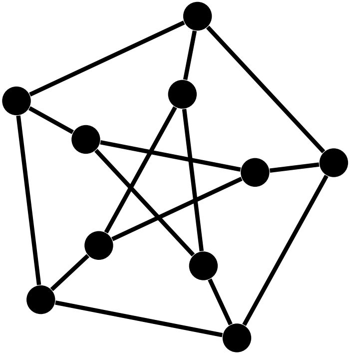

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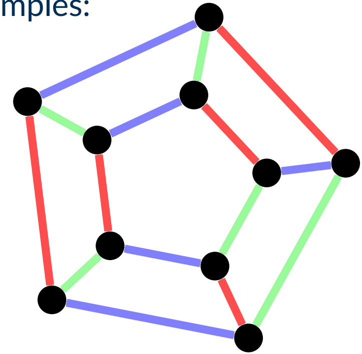

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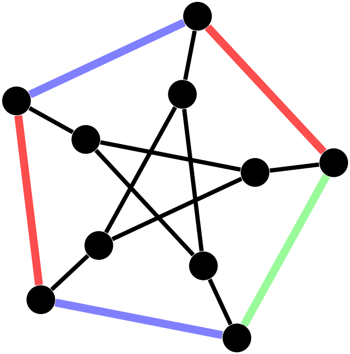

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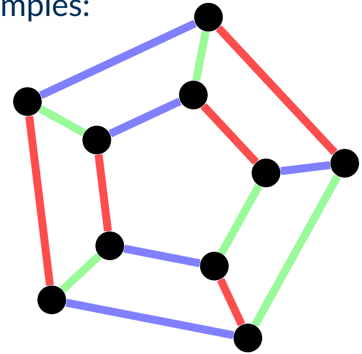

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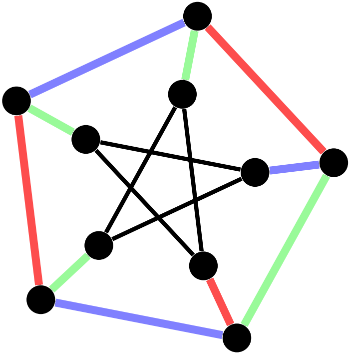

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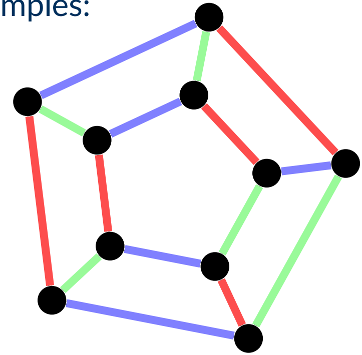

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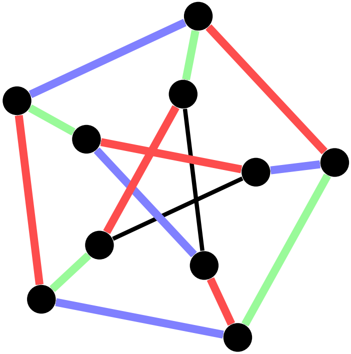

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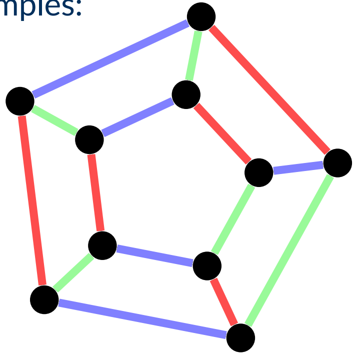

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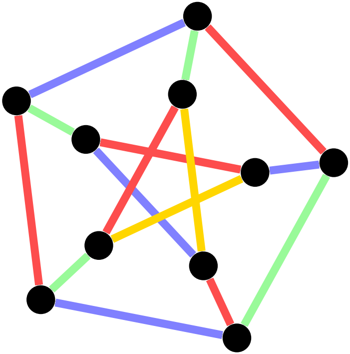

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Approximate Minimum Edge Colorings

Observation.

The maximum node degree $\Delta(G)$ is a lower bound on the required number of colors, i.e.,

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Theorem 2.28 (Vizing 1964)

There is a polynomial-time algorithm (Vizing's Algorithm) that finds a $(\Delta + 1)$-edge-coloring of a graph. In particular,

$$\chi'(G) \in \{\Delta(G), \Delta(G) + 1\} \ .$$

Outline of Vizing's Algorithm

Input: Undirected graph $G = (V, E)$ without parallel edges.

Output: A $(\Delta(G) + 1)$-edge-coloring.

Main idea: Color one new edge in each iteration;
always maintain a feasible partial $(\Delta + 1)$-edge-coloring.

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Useful fact: For any node $v \in V$ there is always a color c that is currently not being used by its incident edges. We say that "v lacks color c."