Approximation Algorithms (ADM III)
3- Rounding Data and Dynamic Programming

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Outline

Knapsack Problem

Given: n items $I = \{1, \ldots, n\}$, values $v_i \in \mathbb{Z}_{>0}$ and sizes $s_i \in \mathbb{Z}_{>0}$, $i \in I$;
knapsack of size $B \in \mathbb{Z}_{>0}$ (assume w.l.o.g. $s_i \leq B$, for all $i \in I$)

Task: find subset of items $S \subseteq I$ with $\sum_{i \in S} s_i \leq B$ maximizing $\sum_{i \in S} v_i$.

KS.png

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Remarks

- The Knapsack Problem is *NP*-hard (reduction from Partition).
- It can be solved in pseudo-polynomial time by dynamic

Dynamic Program for Knapsack Problem (1/3)

- Denote by $J(i,b)$ the maximum value that can be packed in a knapsack of capacity $b \leq B$, using only a subset of items $T \subseteq \{1,\ldots,i\}$:

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- The values of $J(i,b)$ can be computed recursively:

$$J(i,b) = \begin{cases} 0 & \text{if } i = 0; \\ J(i-1,b) & \text{if } s_i > b; \\ \max(J(i-1,b), J(i-1,b-s_i)+v_i) & \text{otherwise.} \end{cases}$$

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- Optimal solution $\texttt{OPT} = J(n,B)$ can be computed by “filling” the table. Hence, the complexity of this algorithm is $O(nB)$.

- However, this is not a polytime algorithm, as the input can be described with only $\langle B \rangle := \log_2 B$ bits.

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- In fact, we can make the best of both worlds and get a DP of complexity $O(n \min(B, V))$, by using a notion of dominance.
- We identify a subset of items T with a pair $(t, w) = (s(T), v(T))$. We say that a pair (t_1, w_1) is dominated by (t_2, w_2) if $t_1 \geq t_2$ and $w_1 \leq w_2$, and $(t_1, w_1) \neq (t_2, w_2)$.

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Remark: An optimal subset $S \subseteq I$ can be obtained by backtracking.

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Proof:… □

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Scheduling Jobs on Identical Parallel Machines

Given: n jobs $j = 1, \dots, n$ with processing times $p_j \geq 0$, $j = 1, \dots, n$, and m identical parallel machines.

Task: Process each job j nonpreemptively for p_j time units on one of the m machines. Minimize the makespan.

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Remember:

- List scheduling in arbitrary order is $\left(2 - \dfrac{1}{m}\right)$-approximation algorithm.
- List scheduling in LPT order is a $\left(\dfrac{4}{3} - \dfrac{1}{3m}\right)$-approximation algorithm.
- The analysis of both results relies on the fact that

$$C_{\max} \leq \left(1 - \tfrac{1}{m}\right) p_\ell + \tfrac{1}{m} \sum_{j=1}^{n} p_j \leq \left(1 - \tfrac{1}{m}\right) p_\ell + C^*_{\max}$$

  where ℓ is a job with maximal completion time $C_\ell = C_{\max}$.

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Theorem 3.4

Algorithm A_ε runs in polynomial time for the problem $Pm||C_{\max}$ and produces a schedule of makespan at most $(1+\varepsilon) \cdot C^*_{\max}$.

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Theorem 3.6

There is a polynomial-time approximation scheme for $P||C_{\max}$.

Existence of an FPTAS

We state the next theorems without proof:

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Remark: More generally, a strongly NP-hard optimization problem whose objective function values are integral and polynomially bounded in the numbers occurring in the input does not have an FPTAS, unless $P = NP$.

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Theorem 3.10

Algorithm Next-Fit runs in $O(n)$ time and uses at most $2 \cdot \mathtt{OPT} - 1$ bins.

First-Fit Heuristics for Bin-Packing

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Algorithm First-Fit runs in polynomial time; it uses at most $\left\lfloor \frac{17}{10} \mathtt{OPT} \right\rfloor$ bins.

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First-Fit Heuristics for Bin-Packing

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Theorem 3.12 (Dósa 2007)

Algorithm First-Fit-Decreasing uses at most $\frac{11}{9}\mathtt{OPT} + \frac{2}{3}$ bins.

Towards an Asymptotic PTAS for Bin-Packing

Definition 3.13

An asymptotic polynomial-time approximation scheme (APTAS) is a family of polynomial-time algorithms $(A_\varepsilon)_{\varepsilon>0}$ along with a constant c such that A_ε returns a solution of value at most $(1+\varepsilon)\texttt{OPT} + c$.

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Remarks:

- For $\gamma=\varepsilon/2$, the lemma yields a packing of all items into at most $\max\left\{\ell,(1+\varepsilon)\texttt{OPT}+1\right\}$ bins.
- In the following we can thus restrict to items of size at least $\varepsilon/2$.

Linear Grouping Scheme

For given instance I and parameter $k \in \mathbb{Z}_{>0}$, define a new instance I':

1. sort the items of instance I such that $a_1 \geq a_2 \geq \cdots \geq a_n$;
2. instance I' has $n - k$ items of size $a'_i := a_{\lceil i/k \rceil \cdot k+1}$, $i = 1, \ldots, n-k$;

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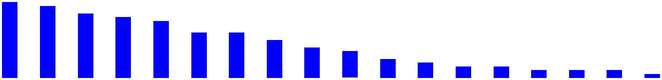

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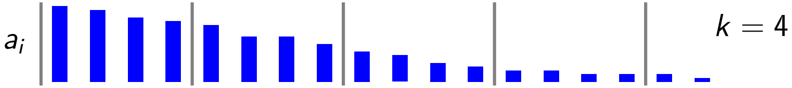

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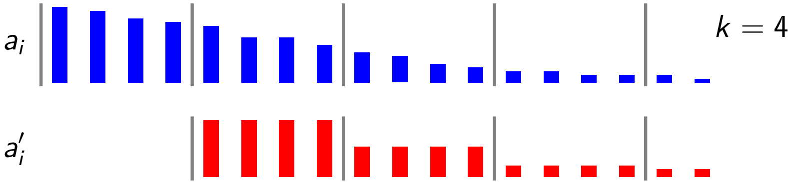

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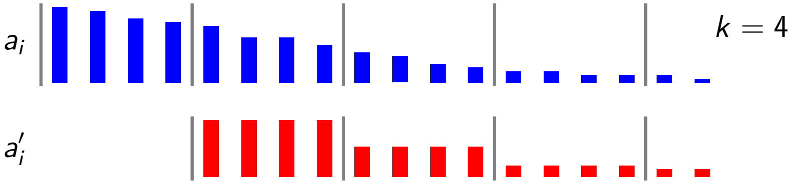

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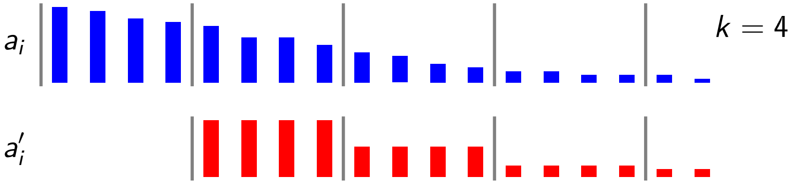

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Lemma 3.15

Any packing of I' can be easily turned into a packing of I with at most k additional bins. Moreover, $\texttt{OPT}(I') \leq \texttt{OPT}(I) \leq \texttt{OPT}(I') + k$.

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