Approximation Algorithms (ADM III)

4- Deterministic Rounding of Linear Programs

Guillaume Sagnol

Outline

The scheduling problem $1|r_j|\sum_j C_j$

Given: jobs with processing time $p_j > 0$, release date $r_j \geq 0$, $j = 1, \ldots, n$.

Task: Schedule the jobs nonpreemptively on a single machine;

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Proof: Use an exchange argument. □

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For each job $j = 1, \ldots, n$, it holds that $C_j \leq 2 \cdot C_j^P$.

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Lemma 4.4

The completion times C_j of a feasible schedule satisfy the LP constraints.

Scheduling in Order of LP Completion Times

Lemma 4.5

Despite the exponential number of constraints, an optimal solution C^* to the LP relaxation can be computed in polynomial time.

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Given: Graph $G = (V, E)$, root node $r \in V$, edge costs $c_e \geq 0$, $e \in E$, and penalties $\pi_i \geq 0$, $i \in V$.

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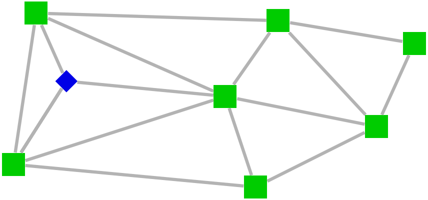

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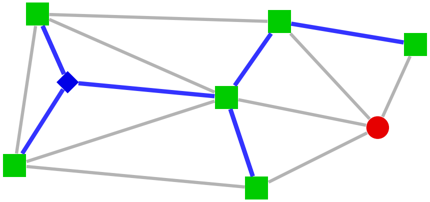

◆ Root node
■ Connected Terminal
● Disonnected Terminal

COST = $c($ —— $) + \pi($ ● $)$

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\text{s.t.} \quad & \sum_{e \in \delta(S)} x_e \geq \max_{i \in S} y_i & \text{for all } S \subseteq V \setminus \{r\}, \\
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LP relaxation: $x_e \geq 0$ for all $e \in E$ and $y_i \leq 1$ for all $i \in V$.

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Theorem 4.8

For $\alpha = 2/3$ the cost of the solution returned by the algorithm is

$$c\big(E(T)\big) + \pi\big(V \setminus V(T)\big) \leq \frac{2}{\alpha}\sum_{e\in E} c_e \cdot x_e^* + \frac{1}{1-\alpha}\sum_{i\in V} \pi_i \cdot (1 - y_i^*) \leq 3 \cdot \mathtt{OPT} \quad .$$

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Interpretation of the dual LP:

- v_j is the total amount that client j wants to pay for being served.
- client j might contribute w_{ij} to facility i for being connected to i.

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For each client j, $v_j^* \geq c_{ij}$ for all $i \in N(j)$.

Structure of Optimal LP Solution

Let (x^*, y^*) and (v^*, w^*) be optimal solutions to the primal and dual LP, respectively.

Notation:

- Facility i *neighbors* client j if $x_{ij}^* > 0$; $N(j) := \{i \in F \mid x_{ij}^* > 0\}$.
- $N^2(j) := \{\ell \in D \mid \text{client } \ell \text{ neighbors some facility } i \in N(j)\}$.

Lemma 4.9

If clients $j_1, \ldots, j_k$ have disjoint neighborhoods $N(j_1), \ldots, N(j_k)$, then opening cheapest facility in each neighborhood costs $\leq \sum_{i \in F} f_i \cdot y_i^* \leq \texttt{OPT}$.

Lemma 4.10

For each client j, $v_j^* \geq c_{ij}$ for all $i \in N(j)$.

Proofs: … □

Deterministic LP Rounding Algorithm

```
compute optimal LP solutions (x*, y*) and (v*, w*);
while D ≠ ∅
      choose j := argmin_{j'∈D} v*_{j'} and i := argmin_{i'∈N(j)} f_{i'};
      assign all unassigned clients in N²(j) to facility i;
      set D := D \ N²(j);
```

Deterministic LP Rounding Algorithm

1. compute optimal LP solutions (x^*, y^*) and (v^*, w^*);
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3. choose $j := \text{argmin}_{j' \in D} v^*_{j'}$ and $i := \text{argmin}_{i' \in N(j)} f_{i'}$;
4. assign all unassigned clients in $N^2(j)$ to facility i;
5. set $D := D \setminus N^2(j)$;

Theorem 4.11

The algorithm above is a 4-approximation algorithm.

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Proof:… □

We finally mention the following non-approximability result without proof.

Theorem 4.12

There is no α-approximation algorithm for the metric uncapacitated facility location problem with $\alpha < 1.463$ unless each problem in NP has an $O(n^{O(\log \log n)})$ time algorithm.

Outline

Bin Packing Revisited

In the previous chapter we showed how to find a solution to instance I with at most $(1+\varepsilon)\texttt{OPT}(I)+1$ bins in polynomial time.

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Ingredients:

- Replace dynamic program with integer program plus LP rounding.
- Improved grouping scheme.
- Recursive application of two previous ingredients.

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Ingredients:

- Replace dynamic program with integer program plus LP rounding.
- Improved grouping scheme.
- Recursive application of two previous ingredients.

Notice:
By Lemma 3.14 we can assume that all items have size at least $1/\mathrm{SIZE}(I)$.

Configuration Integer Program for Bin Packing

- let $s_1 > s_2 > \cdots > s_m$ denote the different item sizes;
- for $i = 1, \ldots, m$, let b_i denote the number of items of size s_i;

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\min & \sum_{j=1}^{N} x_j & \\
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Configuration LP and its Dual

Primal:

$$\begin{aligned} \min \quad & \sum_{j=1}^{N} x_j \\ \text{s.t.} \quad & \sum_{j=1}^{N} t_{ij} \cdot x_j \geq b_i && \text{for all } i = 1, \ldots, m, \\ & x_j \geq 0 && \text{for all } j = 1, \ldots, N. \end{aligned}$$

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Dual:

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Notice: $\text{SIZE}(I) \leq \texttt{OPT}_{LP}(I) \leq \texttt{OPT}(I)$

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An LP solution of value at most $\mathrm{OPT}_{LP}(I) + 1$ can be computed in polynomial time.

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Since $y^*/(1+\varepsilon)$ is feasible dual solution, $\sum_{i=1}^{m} b_i \cdot y_i^* \leq (1+\varepsilon)\mathrm{OPT}_{LP}$.

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Since $y^*/(1+\varepsilon)$ is feasible dual solution, $\sum_{i=1}^{m} b_i \cdot y_i^* \leq (1+\varepsilon)\text{OPT}_{LP}$.

Moreover, for $J := \{j \mid \delta_j = 1\}$, vector y^* is optimal solution to

reduced dual:

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Proof of Theorem 4.13 (Cont.)

Consider the corresponding

reduced primal:

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$\bar{y}$ is feasible solution to original primal LP of value at most $\mathrm{OPT}_{LP} + 1$. □

Harmonic Grouping Scheme and Rounding

Grouping

- consider items in order of non-increasing size;
- open a group and start putting items in current group, one at a time;
- close current group if its total size is at least 2 and start new group;

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Notice that $r \leq \lceil \text{SIZE}(I)/2 \rceil$ and $n_i \geq n_{i-1}$, for $i = 2, \dots, r-1$.

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Rounding: Construct new instance I' as follows:

- discard items in G_1 and G_r;
- for $i = 2, \ldots, r-1$ discard the $n_i - n_{i-1}$ smallest items in G_i;
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Lemma 4.14

There are at most $\text{SIZE}(I)/2$ distinct item sizes in I'; the total size of all discarded items is $O(\log \text{SIZE}(I))$. □

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BinPack(I)

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6. call the packed items instance I_1 and the remaining items I_2;
7. pack I_2 recursively via BinPack(I_2);

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Lemma 4.15

$\mathtt{OPT}_{LP}(I_1) + \mathtt{OPT}_{LP}(I_2) \leq \mathtt{OPT}_{LP}(I') \leq \mathtt{OPT}_{LP}(I).$

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Algorithm BinPack runs in polynomial time and finds a solution using at most $\mathtt{OPT}(I) + O\left(\log^2 \mathtt{OPT}(I)\right)$ bins.

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Proof:… □

Theorem 4.17 (Hoberg & Rothvoß, 2015)

A solution using at most $\mathtt{OPT}(I) + O\left(\log \mathtt{OPT}(I)\right)$ bins can be found in polynomial time. □