Approximation Algorithms (ADM III)

6- The primal dual method

Guillaume Sagnol

Outline

Set Cover Problem

Given: A set of elements $E = \{e_1, \ldots, e_n\}$, a family of subsets $\{S_1, \ldots, S_m\} \subseteq 2^E$, and a weight $w_j \geq 0$ for each $j \in \{1, \ldots, m\}$.

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$$\begin{array}{llll} \min & \displaystyle\sum_{j=1}^{m} w_j \cdot x_j & & \\ \text{s.t.} & \displaystyle\sum_{j:e_i \in S_j} x_j \geq 1 & & \text{for all } i = 1, \ldots, n \\ & x_j \geq 0 & & \text{for all } j = 1, \ldots, m \end{array}$$

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Dual LP:

$$\begin{array}{llcll} \max & \sum_{i=1}^{n} y_i & & & \\ \text{s.t.} & \sum_{e_i \in S_j} y_i & \leq & w_j & \text{for all } j = 1, \dots, m \\ & y_i & \geq & 0 & \text{for all } i = 1, \dots, n \end{array}$$

Primal-Dual Algorithm Set Cover (see Ch. 1)

set $y :\equiv 0$ and $I := \emptyset$;

while $\exists e_k \notin \bigcup_{j \in I} S_j$

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Theorem 6.1 (recap of Theorem 1.10)

The primal-dual algorithm is an f-approximation algorithm for the Set Cover Problem where $f := \max_{i=1,\ldots,n} |\{j \mid e_i \in S_j\}|$.

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Proof:

$$\sum_{j \in I} w_j = \sum_{j \in I} \sum_{i: e_i \in S_j} y_i = \sum_{i=1}^{n} y_i \cdot |\{j \in I \mid e_i \in S_j\}|$$

$$\leq f \cdot \sum_{i=1}^{n} y_i \leq f \cdot \mathtt{OPT}_{LP} \leq f \cdot \mathtt{OPT}$$

□

Approximate Complementary Slackness

Remark. The pair of feasible solutions (x, y) to the primal and dual LP found by the algorithm satisfies

$$x_j > 0 \quad \Longrightarrow \quad \sum_{e_i \in S_j} y_i = w_j \qquad (\textit{compl. slackness})$$

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The analysis on the previous slide only relies on these two properties!

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Feedback Vertex Set Problem

Given: Undirected graph $G = (V, E)$ with node weights $w_i \geq 0$, $i \in V$.

Task: Find $S \subseteq V$ minimizing $\sum_{i \in S} w_i$ such that $G[V \setminus S]$ is acyclic.

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Integer programming formulation: Let $\mathcal{C}$ denote the set of all cycles in G.

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Dual of LP relaxation ($x \geq 0$):

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- But: If we choose arbitrary C in each iteration, $|S \cap C|$ can be large.
- Idea: Always choose short cycle C with $|C| \leq \alpha$.
- But: This is not always possible (e.g., if graph is one large cycle).
- Obs.: From path of nodes of degree two, algorithm chooses ≤ 1 node.

Refined Primal-Dual Algorithm for Feedback Vertex Set

Lemma 6.2

In any graph G that has no nodes of degree one, there is a cycle with $\leq 2\lceil \log_2 n \rceil$ nodes of degree 3 or more, and it can be found in linear time.

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Proof:… □

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If the primal-dual algorithm chooses in each iteration a cycle with at most $\leq 2\lceil \log_2 n \rceil$ nodes of degree 3 or more, it has performance ratio $4\lceil \log_2 n \rceil$.

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Proof:… □

Remarks.

- The LP relaxation has an integrality gap of $\Omega(\log n)$.
- There is a primal-dual 2-approximation algorithm based on a more sophisticated integer programming formulation

Outline

Shortest s-t-Path Problem

Given: Undir. graph $G = (V, E)$ with edge costs $c_e \geq 0$, $e \in E$; $s, t \in V$

Task: Find minimum-cost s-t-path.

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Dual of LP relaxation ($x \geq 0$):

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Primal-Dual Algorithm for Shortest s-t-Path Problem

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set y := 0 and F := ∅;
while there is no s-t-path in F
    let C be the connected component of (V, F) containing s;
    increase y_C until there is an e ∈ δ(C) with ∑_{S∈𝒮: e∈δ(S)} y_S = c_e;
    set F := F ∪ {e};
delete edges from F that do not lie on s-t-path in F;
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Lemma 6.4

Throughout the algorithm, the set of edges in F always forms a tree containing node s. □

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Theorem 6.5

The algorithm finds a shortest s-t-path.

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Given: Graph $G = (V, E)$ with costs $c_e \geq 0$, $e \in E$; k pairs $s_i, t_i \in V$.

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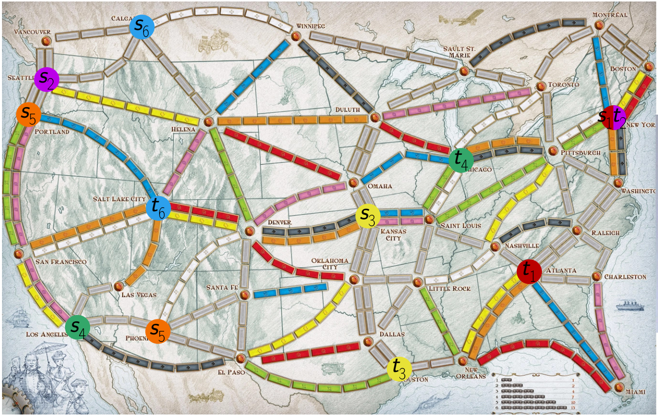

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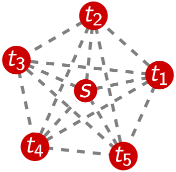

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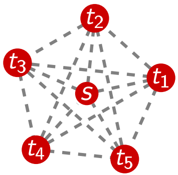

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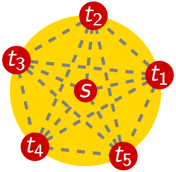

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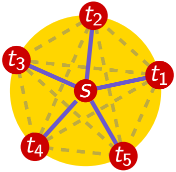
Analysis: $$\sum_{e \in F} c_e = \sum_{e \in F} \sum_{S \in \mathcal{S}:\, e \in \delta(S)} y_S = \sum_{S \in \mathcal{S}} |\delta(S) \cap F| \cdot y_S$$

Problem: It can happen that $|\delta(S) \cap F| = k$ for $y_S > 0$ and $\sum_{S \in \mathcal{S}} y_S \leq \frac{1}{k}\mathtt{OPT}$:

- $G = K_{k+1}$ (complete graph)
- $s_i := s$ for $i = 1, \ldots, k$
- $c_e := 1$ for all $e \in E$



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Primal-Dual Algorithm for Steiner Forest Problem

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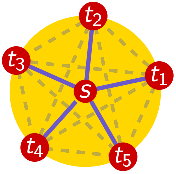

- $y_{\{s\}} = 1$
- $|\delta(\{s\}) \cap F| = k$
- $\sum_S y_S = 1$, $\texttt{OPT} = k$

Refined Primal-Dual Algo for Steiner Forest Problem

set $y :\equiv 0$ and $F := \emptyset$;

while not all s_i-t_i pairs are connected in (V, F)

  let $\mathcal{C} := \{$conn. comp. C of (V, F): $|C \cap \{s_i, t_i\}| = 1$ for some $i\}$;

  increase y_C for all $C \in \mathcal{C}$ uniformly until for some $e \in \delta(C)$, $C \in \mathcal{C}$

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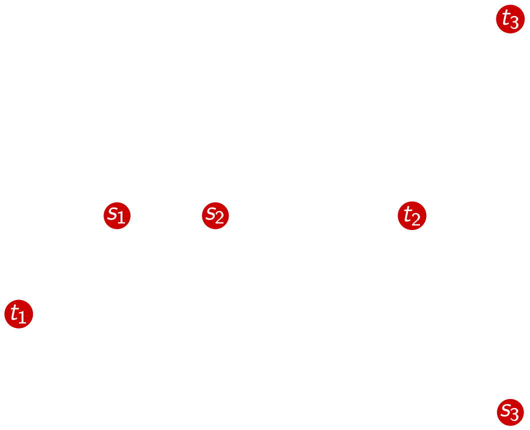
    set $F := F \cup \{e\}$;

For all $e \in F$ (in reverse of the order in which they were added)

    if $F \setminus \{e\}$ is a feasible solution, then remove e from F;

Example



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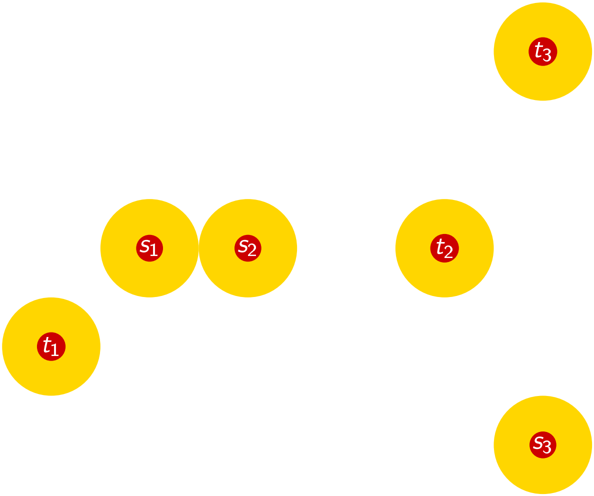

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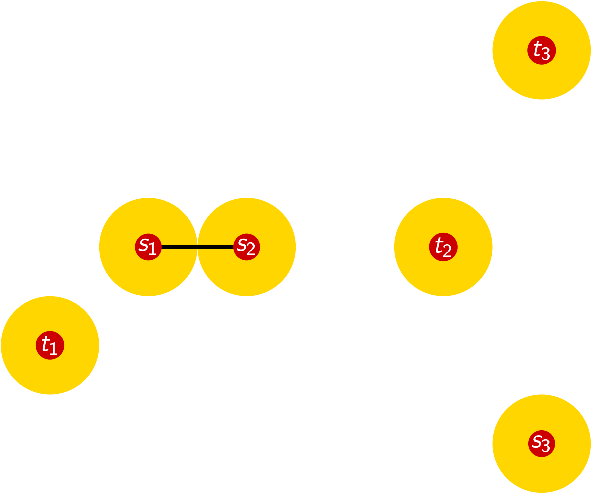

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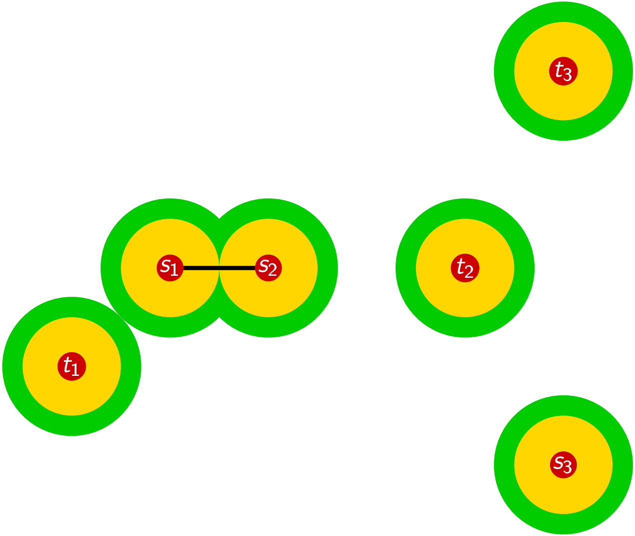

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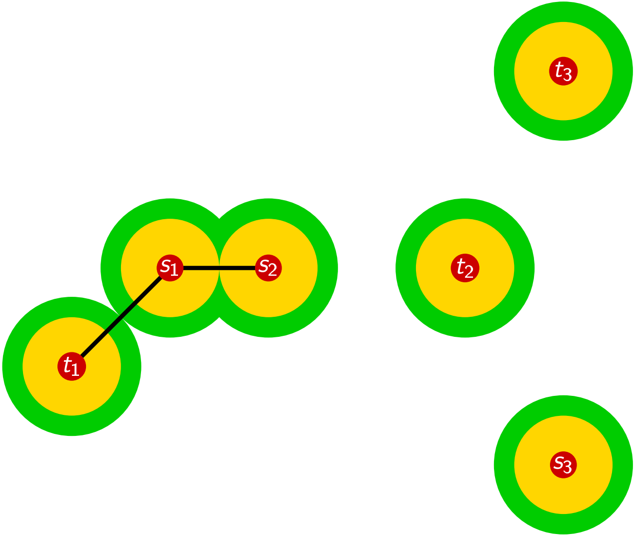

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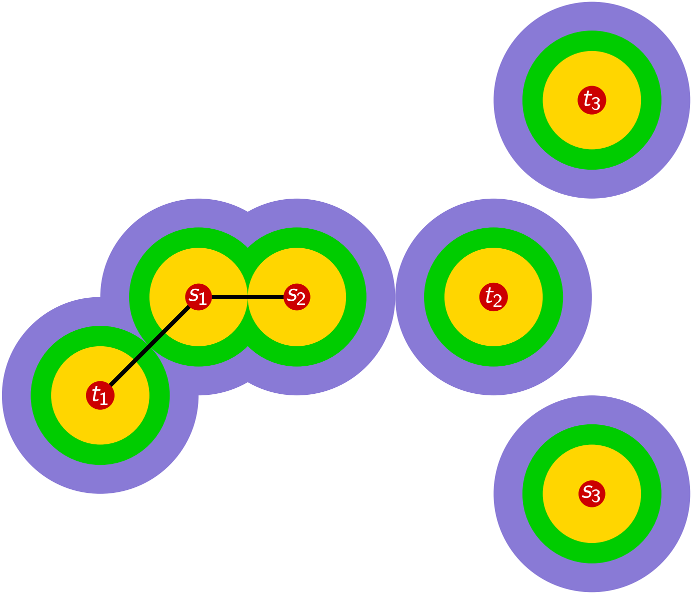

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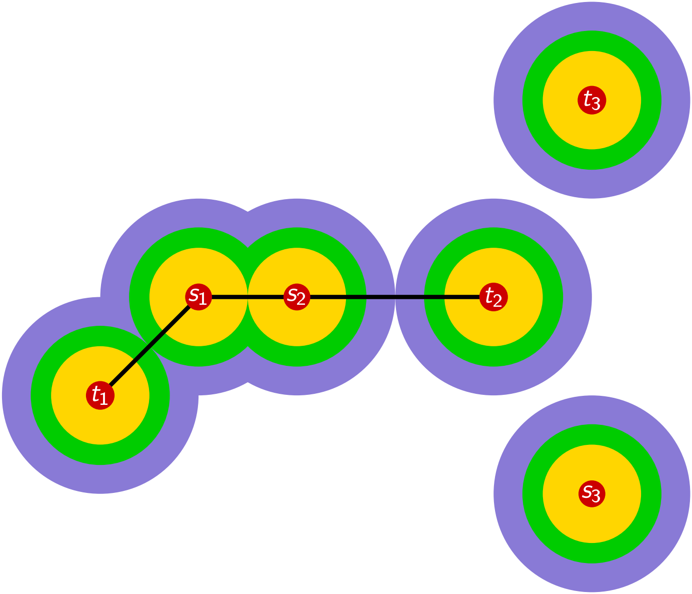

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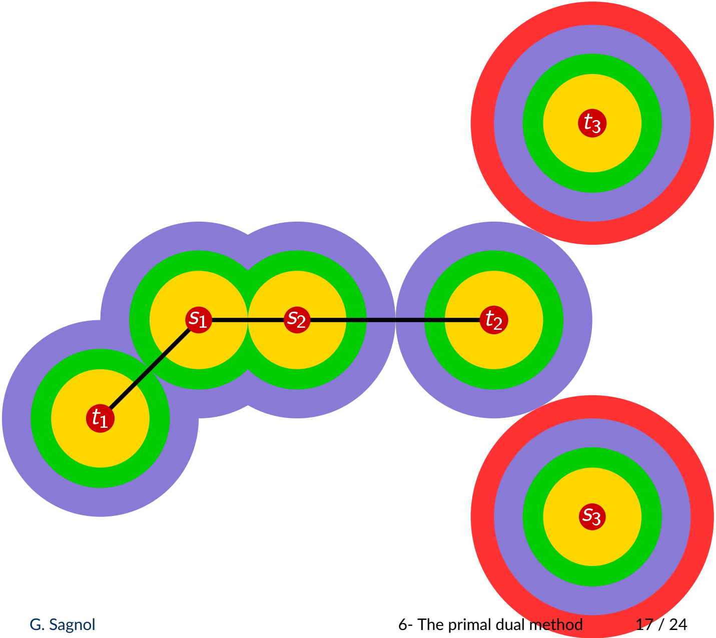

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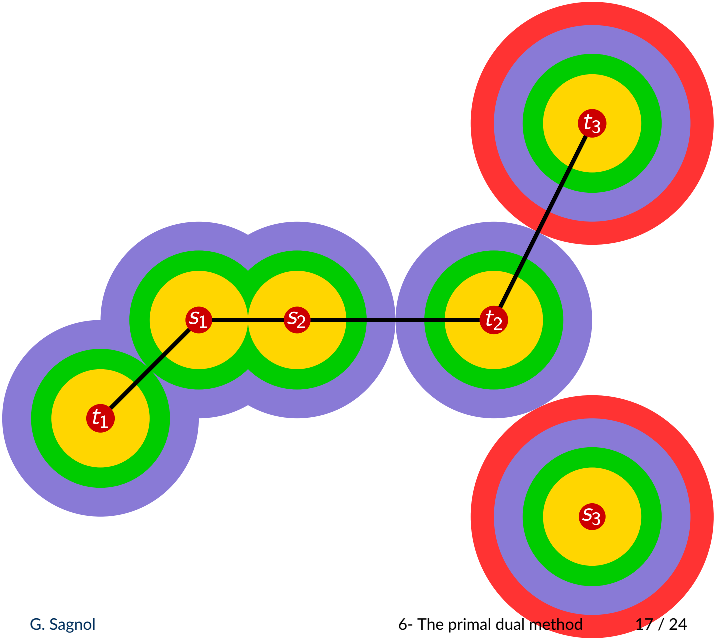

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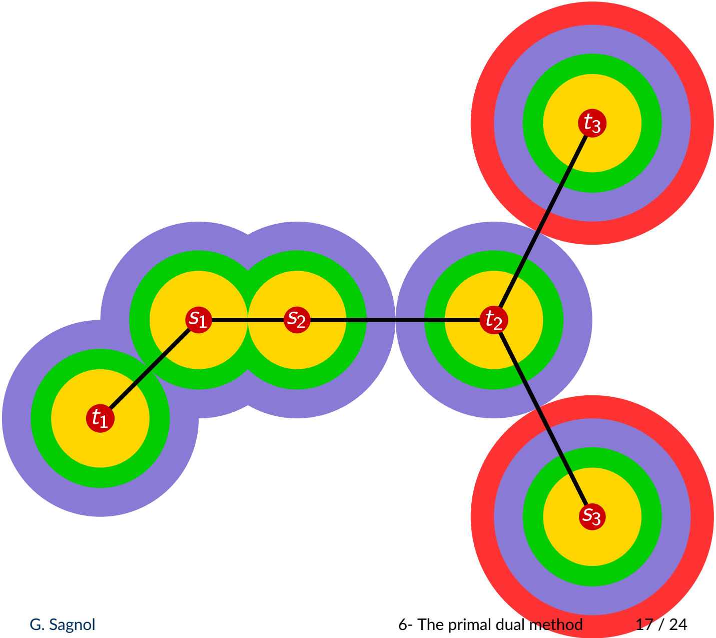

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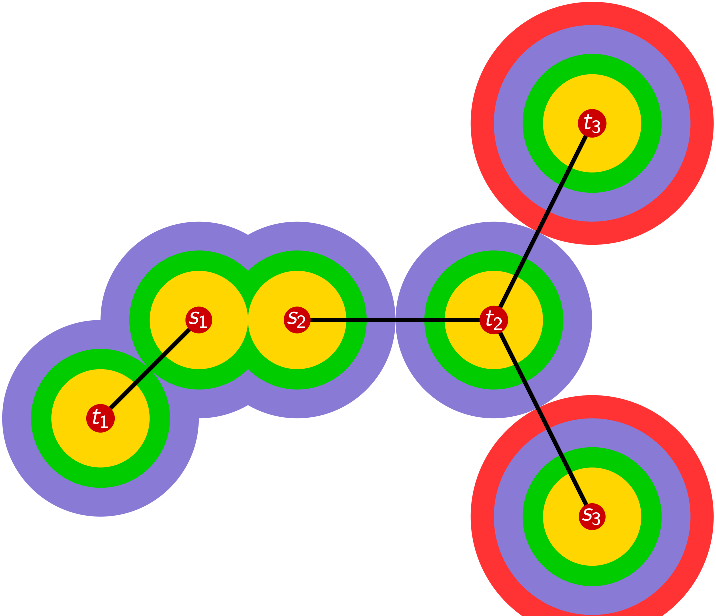

Analysis

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Proof:… □

Corollary 6.8

Integrality gap of the LP relaxation is at most 2. This bound is tight. □

Outline

Uncapacitated Facility Location Problem

Given: Set of facilities F with opening costs $f_i \geq 0$, $i \in F$;
set of clients D with metric connection costs $c_{ij} \geq 0$, $i \in F, j \in D$.

Task: Choose $F' \subseteq F$ and assign each client to nearest facility in F'.

Objective: Minimize $\sum_{i \in F'} f_i + \sum_{j \in D} \min_{i \in F'} c_{ij}$.

IP formulation:

$$\begin{aligned}
\min \quad & \sum_{i \in F} f_i \cdot y_i + \sum_{i \in F, j \in D} c_{ij} \cdot x_{ij} & \\
\text{s.t.} \quad & \sum_{i \in F} x_{ij} = 1 & \text{for all } j \in D, \\
& y_i - x_{ij} \geq 0 & \text{for all } i \in F, j \in D, \\
& x_{ij}, y_i \in \{0, 1\} & \text{for all } i \in F, j \in D.
\end{aligned}$$

LP Relaxation and Dual LP

primal LP:

$$\begin{aligned} \min \quad & \sum_{i \in F} f_i \cdot y_i + \sum_{i \in F, j \in D} c_{ij} \cdot x_{ij} & \\ \text{s.t.} \quad & \sum_{i \in F} x_{ij} = 1 & \text{for all } j \in D, \\ & y_i - x_{ij} \geq 0 & \text{for all } i \in F, j \in D, \\ & x_{ij}, y_i \geq 0 & \text{for all } i \in F, j \in D. \end{aligned}$$

dual LP:

$$\begin{aligned} \max \quad & \textstyle\sum_{j \in D} v_j & \\ \text{s.t.} \quad & \textstyle\sum_{j \in D} w_{ij} \leq f_i & \text{for all } i \in F, \\ & v_j - w_{ij} \leq c_{ij} & \text{for all } i \in F, j \in D, \\ & w_{ij} \geq 0 & \text{for all } i \in F, j \in D. \end{aligned}$$

Interpretation of dual LP:

- v_j is total amount that client j wants to pay for being served.

LP Relaxation and Dual LP

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Interpretation of dual LP:

- v_j is total amount that client j wants to pay for being served.
- client j might contribute w_{ij} to facility i for being connected to i.

Primal-Dual Algo for Uncapacitated Facility Location

Notation & High-level idea:

- For the current feasible dual solution (v, w) and $j \in D$ let
$$N(j) := \{i \in F \mid v_j \geq c_{ij}\} \ .$$

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- During the algorithm, we increase the v_j's ans w_{ij}'s, in a way that
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  - client j contributes only to facilities $i \in N(j)$;
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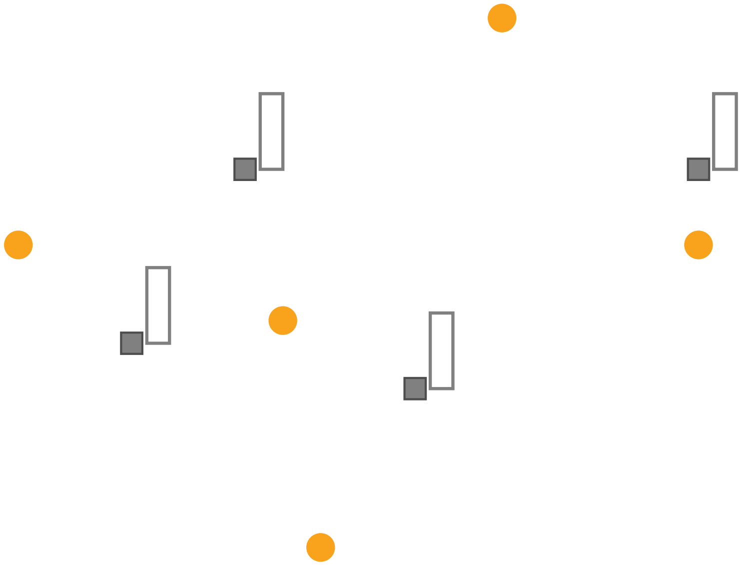
- The set S is the set of clients not yet in the neighborhood of a tight facility; We'll iterate until $S = \emptyset$.

Primal-Dual Algo for Uncapacitated Facility Location

Algorithm:

1. set $S := D$, $T := \emptyset$, $v_j := 0$, $w_{ij} := 0$ for all $i \in F, j \in D$;
2. while $S \neq \emptyset$
3. for all $j \in S$ increase v_j and w_{ij} for all $i \in N(j)$ uniformly until one of the following occurs:
   - i. $\sum_{j \in D} w_{ij} = f_i$ for some $i \notin T$; Then, $T \leftarrow T \cup \{i\}$
   - ii. $v_j \geq c_{ij}$ for some $i \in T$, $j \in S$; Then, $S \leftarrow S \setminus \{j\}$
4. set $F' := \emptyset$
5. while $T \neq \emptyset$ pick $i \in T$;
6. $F' := F' \cup \{i\}$; $T := T \setminus \{h \mid \exists j \in D, \, w_{ij} > 0, \, w_{hj} > 0\}$;
7. open all facilities in F'; connect each $j \in D$ to nearest $i \in F'$.

Example



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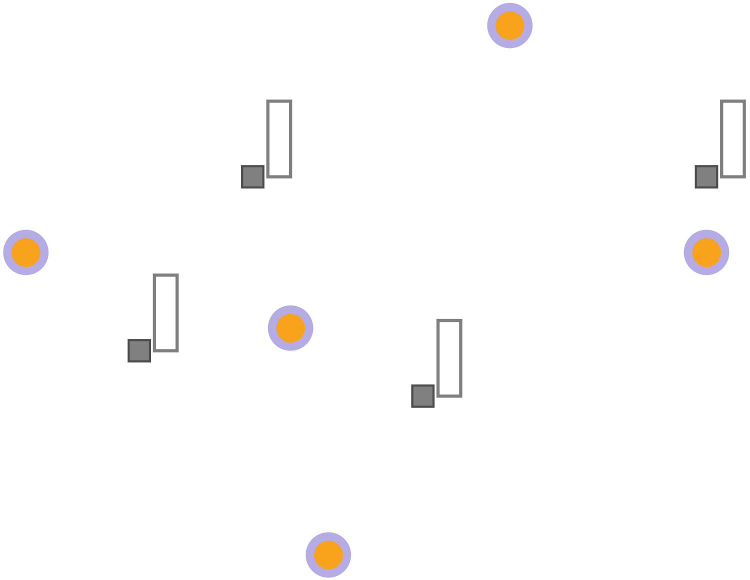

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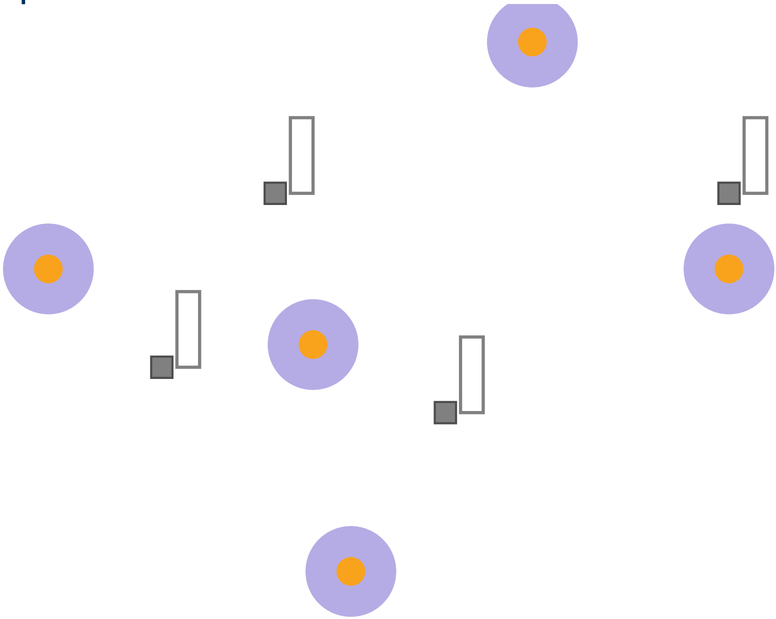

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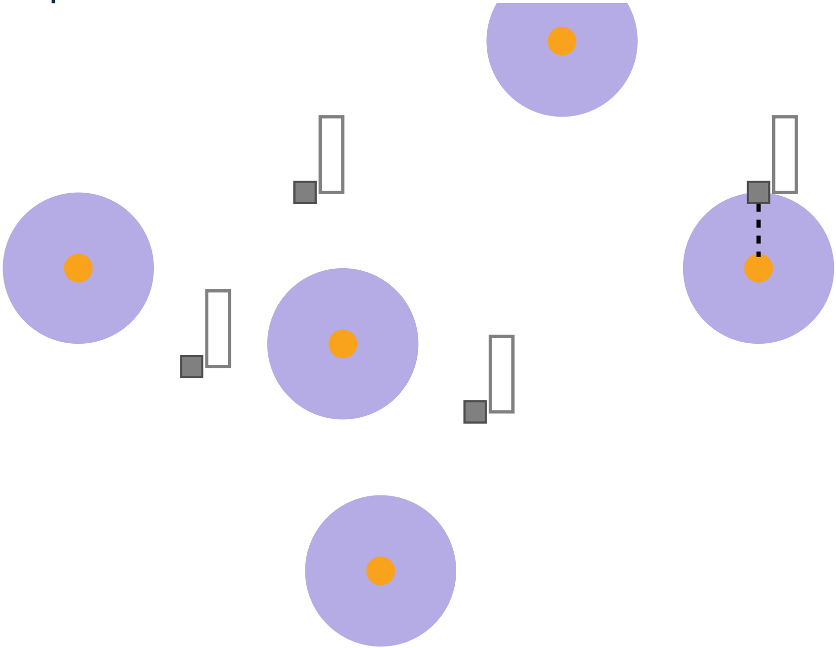

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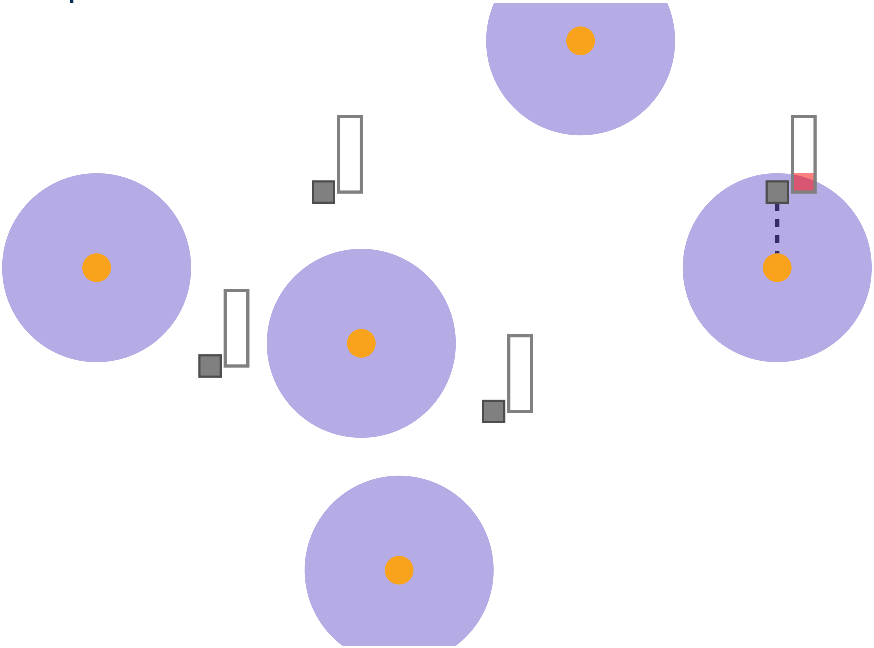

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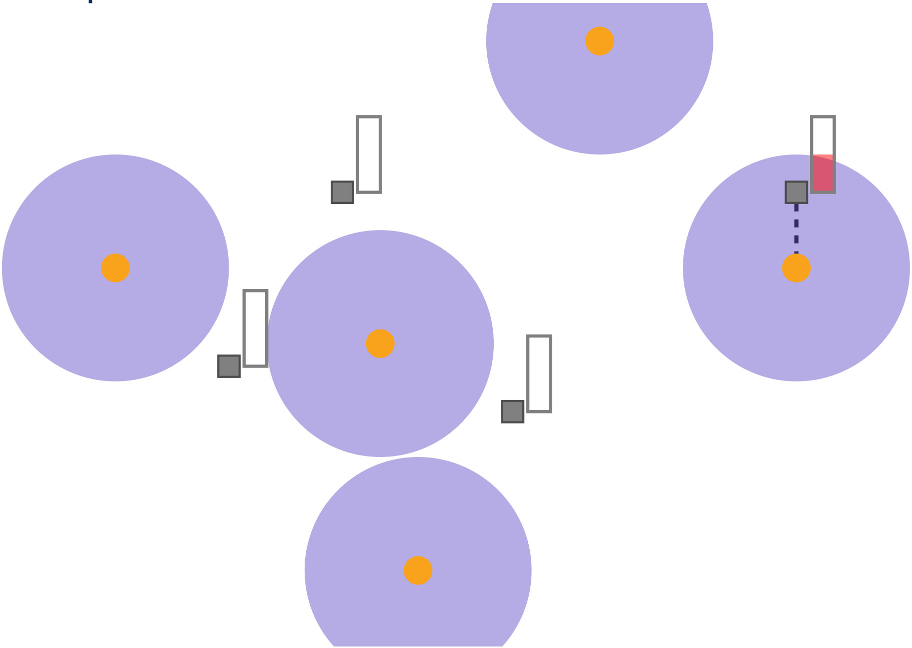

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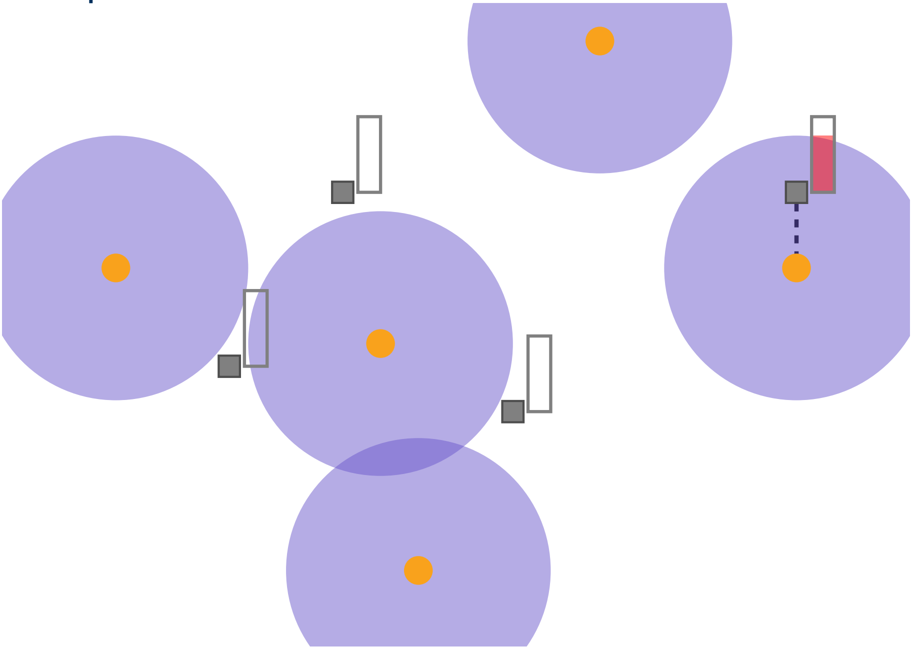

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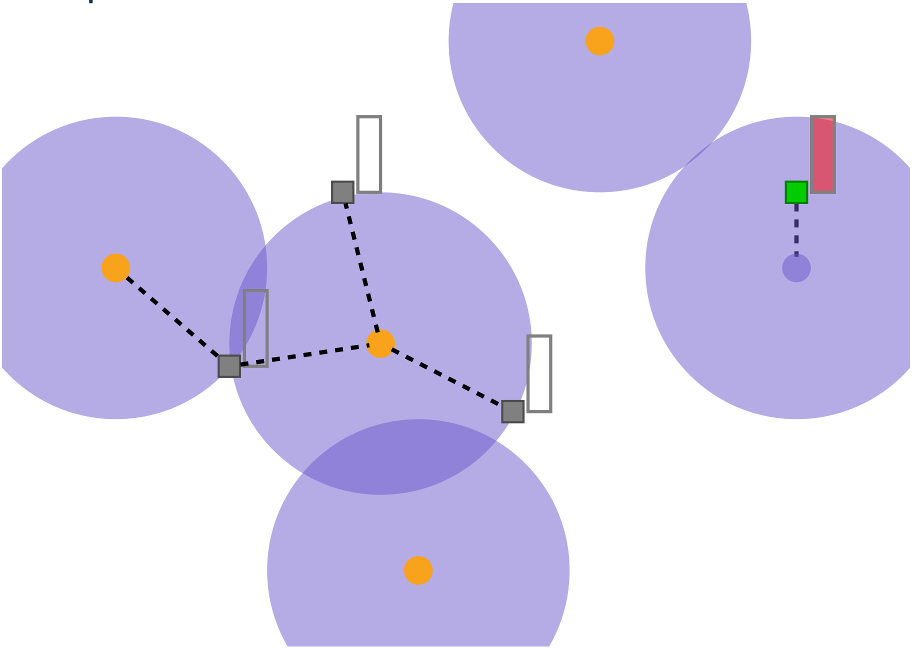

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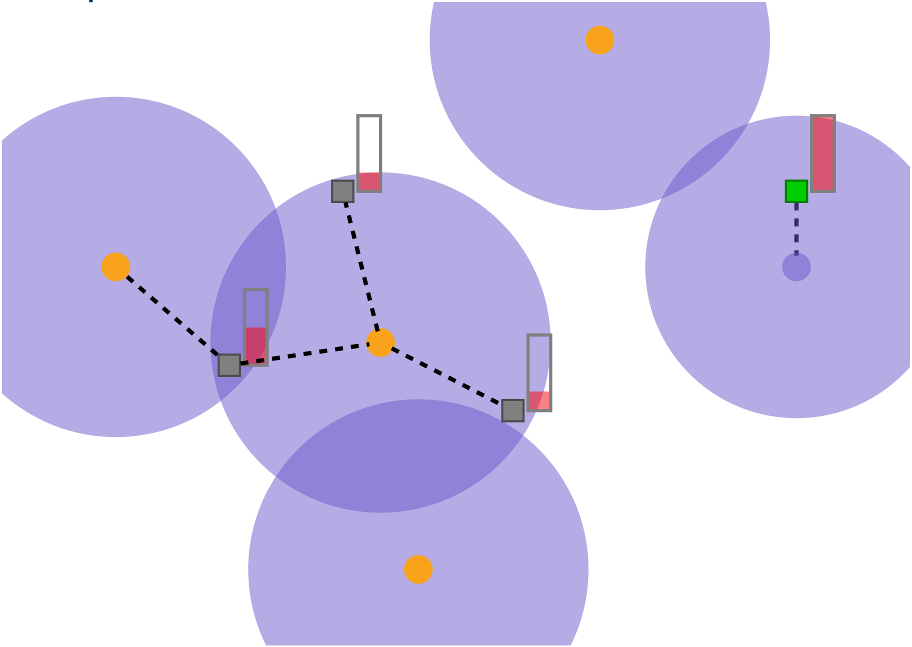

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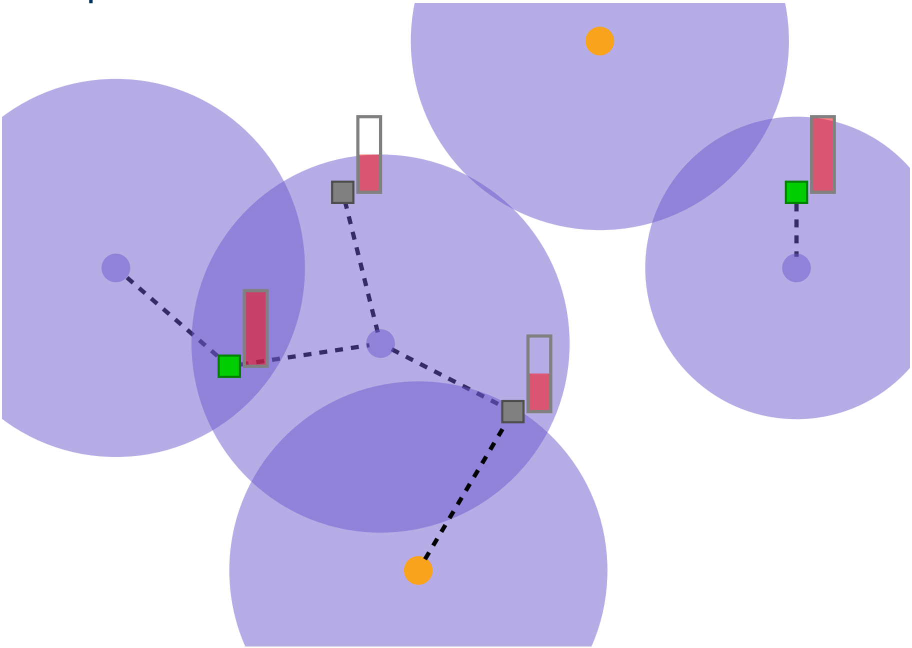

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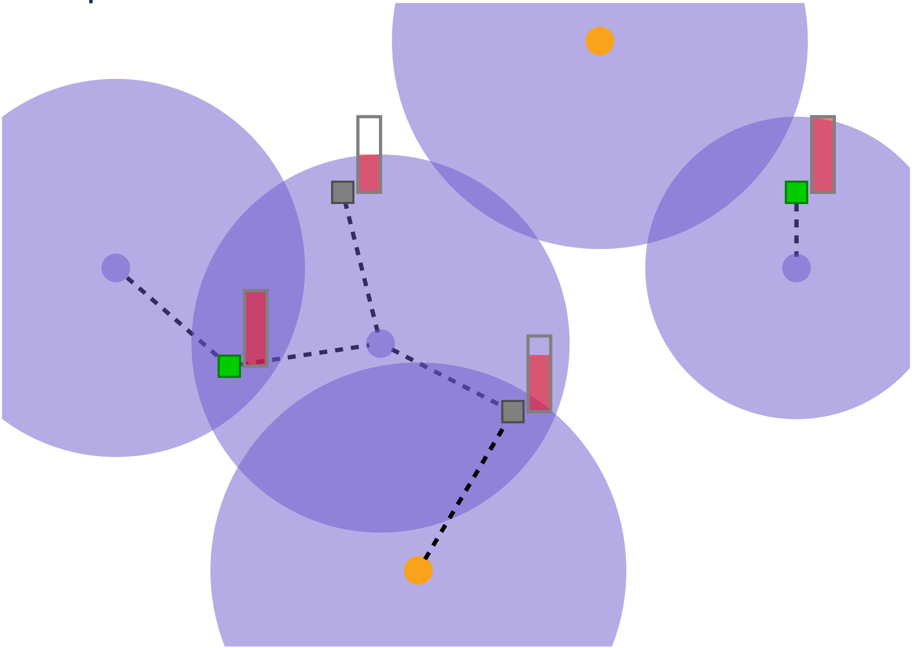

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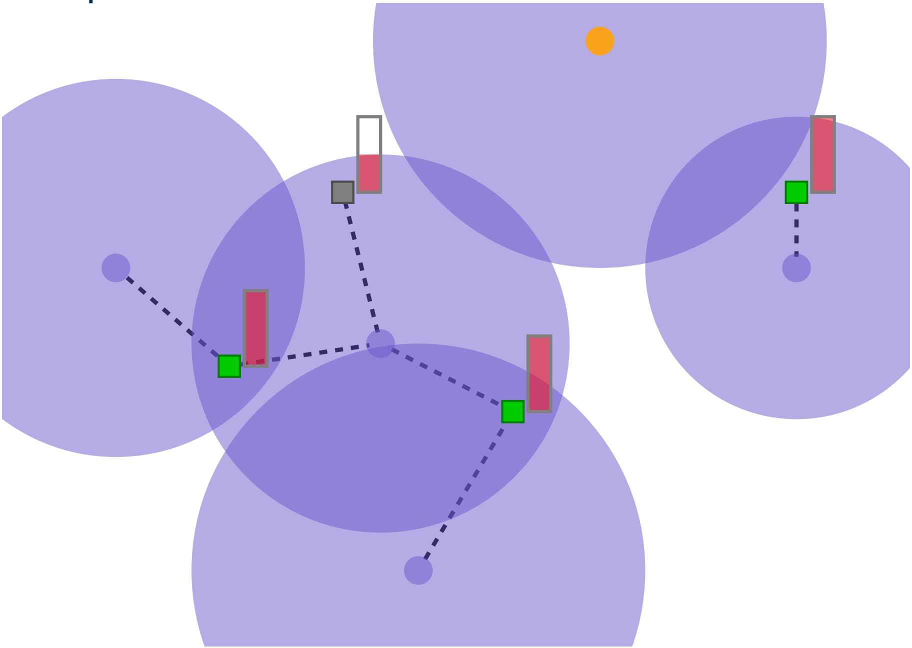

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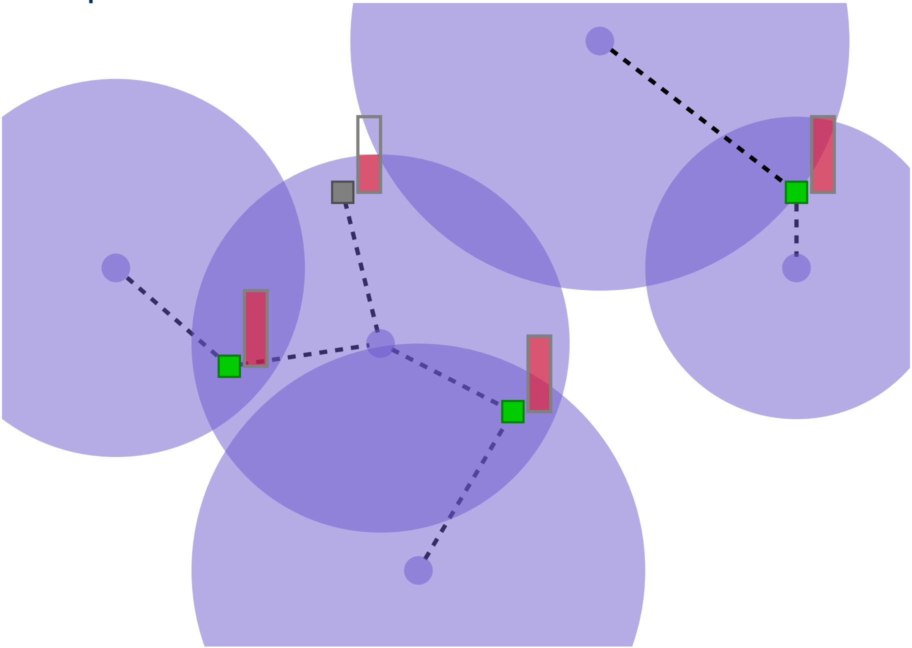

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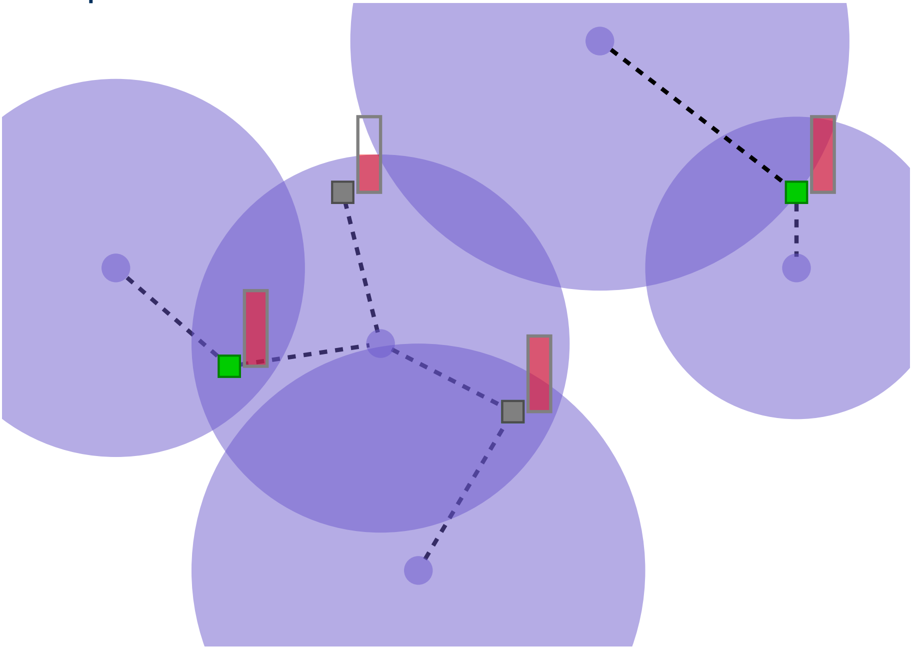

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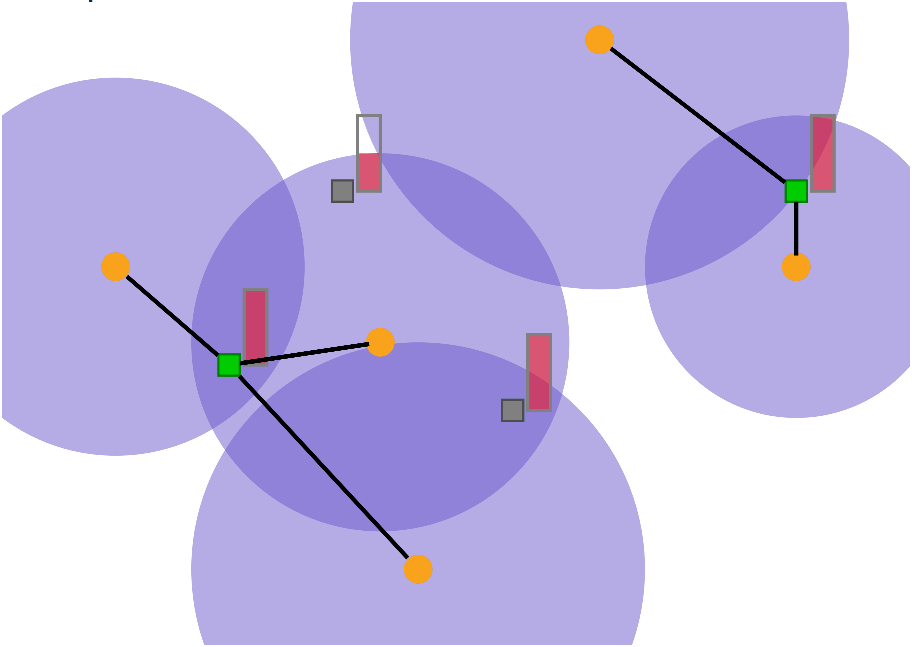

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