Lecture #10 Notes Summary

Reducing the PoA: pricing, augmenting capacities, and Stackelberg routing.

Summary of important points on PoA

- Continuous (Non-atomic) Congestion games
  - Social cost defined as $C(\boldsymbol{x}) = \sum_e x_e d_e(x_e)$ over the set of feasible loads X.
  - A Social optimum is a flow $(\boldsymbol{f^o}, \boldsymbol{x^o})$, such that $\boldsymbol{x^o}$ minimizes $C(\boldsymbol{x})$ over X.
  - A flow $(\boldsymbol{f^*}, \boldsymbol{x^*})$ is a Wardrop Equilibrium iff it minimizes $\Phi(\boldsymbol{x}) = \sum_e \int_0^{x_e} d_e(z)dz$ over X.
  - $PoA = \frac{C(\boldsymbol{x^*})}{C(\boldsymbol{x^o})}$, independently of the choice of a WE $\boldsymbol{x^*}$ or a social optimum $\boldsymbol{x^o}$.
  - For linear delays, $PoA \leq 4/3$. For polynomial delays of degree $\leq p$, $PoA \leq O(p/\log(p))$.
- Discrete (atomic) congestion games
  - Social cost defined as before.
  - For unweighted games, and games with linear delays, there is a potential function $\Phi : \boldsymbol{S} \to \mathbb{R}$, such that every local minimum of Φ is a Nash equilibrium.
  - If X_{eq} is the set of loads induced by a Nash Equilibrium profile and $\boldsymbol{x^o}$ is a social optimum, we defined
  $$PoA = \frac{\max_{\boldsymbol{x} \in X_{eq}} C(\boldsymbol{x})}{C(\boldsymbol{x^o})}, \qquad PoS = \frac{\min_{\boldsymbol{x} \in X_{eq}} C(\boldsymbol{x})}{C(\boldsymbol{x^o})}.$$
  - For linear delays, $PoA \leq \phi^2 = \frac{3+\sqrt{5}}{2}$.
  - In an unweighted game with linear delays, $PoS \leq 1 + 1/\sqrt{3}$.

Reducing the Price of Anarchy

We consider only continuous (non-atomic) congestion games. For the remaining of the lecture we take a generic NACG $\mathcal{G} = (E, \mathcal{S}, \boldsymbol{w}, \boldsymbol{d})$. We next review three techniques to reduce the PoA of a given instance.

1. Pricing

In this section we assume that a toll τ_e has been set on each element $e \in E$, and that the users have to pay a cost $d_e^{\boldsymbol{\tau}}(x_e) := d_e(x_e) + \tau_e$ to take element e. So under this assumption, it is natural to claim that users will behave as in a Wardrop Equilibrium of the game $\mathcal{G}^\tau = (E, \mathcal{S}, \boldsymbol{w}, \boldsymbol{d^\tau})$.

A natural question is thus the following: how should we set the toll prices τ_e, if we want to reduce as much as possible the social cost $C(\boldsymbol{x})$. The answer to this question is surprisingly easy:

Theorem 1. *Let $x \to x d_e(x)$ be convex and of class $\mathcal{C}^1$ for all $e \in E$. Let $\boldsymbol{x^o}$ be a social optimum of $\mathcal{G}$, and define $\tau_e := x^o d_e'(x_e^o)$, where d_e' is the derivative of d_e. Then, $\boldsymbol{x^o}$ is a Wardrop Equilibrium of $\mathcal{G}^\tau$.*

Proof. This is a direct corollary of Proposition 4 in Lecture #6 (equivalence between the notions of social optimum for $\mathcal{G}$ and Wardrop equilibrium for the game $\mathcal{G}'$ with marginal costs $\boldsymbol{\hat{d}}$). □

This simple result actually hides many interesting questions. For example, what is the minimum pricing policy that leads to a social optimum, or what happens if the users have a different perception of the toll fee (we should not add *time* with *money*, hence in reality $d_e(x_e)$ must be converted to a monetary value to compute $d_e^{\boldsymbol{\tau}}$. But this is a problem when users are not uniform and have a different perception of "what time costs".)

2. Augmenting capacities

The next theorem shows that a Wardrop equilibrium is always better than the social optimum in the situation with twice as much traffic.

> **Theorem 2.** *Let $\mathcal{G} = (E, \mathcal{S}, \boldsymbol{w}, \boldsymbol{d})$ be a NACG, and let $(\boldsymbol{f}^*, \boldsymbol{x}^*)$ be a WE for this game. If $(\boldsymbol{f}, \boldsymbol{x})$ is any feasible flow of $\mathcal{G}^1 = (E, \mathcal{S}, 2\boldsymbol{w}, \boldsymbol{d})$, then $C(\boldsymbol{x}^*) \leq C(\boldsymbol{x})$. In particular, the WE equilibrium is always better than the social optimum in the situation with twice as much traffic.*

This can be translated into a theorem for the situation in which investments have been done to augment the link capacities:

> **Corollary 3.** *Let $\mathcal{G} = (E, \mathcal{S}, \boldsymbol{w}, \boldsymbol{d})$ be a NACG, and define the delay functions for augmented capacities: $\tilde{d}_e(x) = \frac{1}{2} d_e(x/2)$. Let $(\boldsymbol{f}^*, \boldsymbol{x}^*)$ be a WE of the game $\tilde{\mathcal{G}} = (E, \mathcal{S}, \boldsymbol{w}, \tilde{\boldsymbol{d}})$ with cost $\tilde{C}(\boldsymbol{x}^*) := \sum_e x_e^* \tilde{d}_e(x_e^*)$. Then, for any feasible flow $(\boldsymbol{f}, \boldsymbol{x})$ of $\mathcal{G}$, we have $\tilde{C}(\boldsymbol{x}^*) \leq C(\boldsymbol{x})$.*

These theorems are proved in Exercise 1 of Worksheet #10.

3. Stackelberg Routing

In a Stackelberg game, we assume that a centralized player can route a fraction α of the whole traffic. A Stackelberg strategy consists in a flow $\boldsymbol{g}$ (inducing a load $\boldsymbol{x}^{\boldsymbol{g}}$) satisfying:

$$\forall i \in [N], \quad \sum_{P \in \mathcal{S}^i} g_P := u_i \leq w_i \tag{1}$$

$$\sum_{P \in \mathcal{S}} g_P = \alpha \sum_i w_i. \tag{2}$$

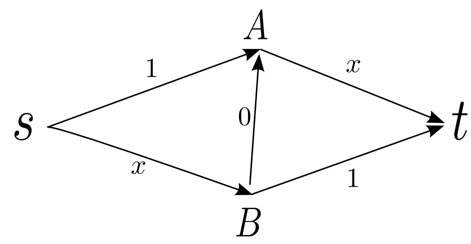
Then, we make the assumption than the remaining fraction $(1 - \alpha)$ of the traffic is routed according to a Wardrop Equilibrium $(\boldsymbol{h}, \boldsymbol{x}^{\boldsymbol{h}})$ of the game where $\boldsymbol{g}$ is fixed, i.e. $\mathcal{G}^{\boldsymbol{g}} = (E, \mathcal{S}, \boldsymbol{w} - \boldsymbol{u}, \boldsymbol{d}^{\boldsymbol{g}})$, where

$$d_e^g(x) := d_e(x_e^g + x).$$

The Price of Anarchy for the Stackelberg strategy $\boldsymbol{g}$ is then defined as $PoA(\boldsymbol{g}) = \frac{C(\boldsymbol{x}^{\boldsymbol{g}} + \boldsymbol{x}^{\boldsymbol{h}})}{C(\boldsymbol{x}^{\boldsymbol{o}})}$, where $\boldsymbol{x}^{\boldsymbol{o}}$ is a social optimum (of the original game $\mathcal{G}$).

The following question has been studied in the literature: Given a fraction α of *controllable* traffic, how must we chose $\boldsymbol{g}$ so that the resulting traffic $\boldsymbol{f} = \boldsymbol{g} + \boldsymbol{h}$ is as close as possible to the social optimum ?

We next give a simple example (from [Bonifaci, Harks & Schäfer, 2010]), where it is possible to compute the optimal Stackelberg strategy: consider the following Braess paradox-like network, where 1 unit of flow must be routed from s to t:



For any feasible flow $\boldsymbol{f}$, we denote by f_1 the amount routed on $s \to A \to t$, by f_2 the amount routed on $s \to B \to t$, and by f_3 the amount of flow routed on the zig-zag strategy $s \to B \to A \to t$. We have the following lemma:

Lemma 4. *Let $\boldsymbol{g}$ be any feasible Stackelberg strategy of weight α for the network above, i.e. $g_1+g_2+g_3 = \alpha$. Then, the unique Wardrop equilibrium $\boldsymbol{h^*}$ of $\mathcal{G}^{\boldsymbol{g}}$ is $h_1^* = h_2^* = 0$, $h_3^* = 1 - \alpha$.*

Proof. Let $\boldsymbol{g}$ be a Stackelberg strategy of weight α ($g_1 + g_2 + g_3 = \alpha$), and $\boldsymbol{h}$ be a feasible strategy for the players not centrally controlled, i.e. $h_1 + h_2 + h_3 = 1 - \alpha$. For a fixed $\boldsymbol{g}$, the costs on each path from s to t are:

$$\begin{aligned} c_1(\boldsymbol{h}) &= 1 + g_1 + g_3 + h_1 + h_3 \\ c_2(\boldsymbol{h}) &= 1 + g_2 + g_3 + h_2 + h_3 \\ c_3(\boldsymbol{h}) &= g_2 + g_3 + h_2 + h_3 + g_1 + g_3 + h_1 + h_3 = 1 + g_3 + h_3, \end{aligned}$$

where we have used that $g_1 + g_2 + g_3 + h_1 + h_2 + h_3 = 1$ to simplify the expression of $c_3(\boldsymbol{h})$. So we have $c_3(\boldsymbol{h}) \leq c_1(\boldsymbol{h})$ and $c_3(\boldsymbol{h}) \leq c_2(\boldsymbol{h})$, which shows that $\boldsymbol{h^*} = [0, 0, 1 - \alpha]^T$ is a Wardrop equilibrium of $\mathcal{G}^{\boldsymbol{g}}$. Moreover, these inequalities are strict if $g_1 > 0$ and $g_2 > 0$, in which case $\boldsymbol{h^*}$ is the only Wardrop equilibrium induced by $\boldsymbol{g}$. If $g_1 = 0$, then we have $c_1(\boldsymbol{h}) = c_3(\boldsymbol{h})$ only in the case where $h_1 = 0$, so we never have an incentive to route drivers on the route 1. A similar argument holds for h_2, so that every WE must satisfy $h_1 = h_2 = 0$ and $\boldsymbol{h^*}$ is the only WE of $\mathcal{G}^{\boldsymbol{g}}$. $\square$

We are thus in a nice situation where the WE flow induced by g does not depend on g. This allows us to compute easily the optimal Stackelberg strategy: for an arbitrary Stackelberg strategy $\boldsymbol{g}$ of weight α, the global flow is

$$\boldsymbol{f} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 + 1 - \alpha \end{pmatrix}.$$

This flow has a cost

$$C(\boldsymbol{x^g} + \boldsymbol{x^h}) = \underbrace{g1}_{s \to A} + \underbrace{g_2}_{B \to t} + \underbrace{(g_1 + g_3 + 1 - \alpha)^2}_{A \to t} + \underbrace{(g_2 + g_3 + 1 - \alpha)^2}_{s \to B},$$

must be minimized under the constraint $g_1+g_2+g_3 = \alpha$ (and $\boldsymbol{g} \geq \boldsymbol{0}$). Using this inequality, the cost rewrites

$$C(\boldsymbol{x^g} + \boldsymbol{x^h}) = g_1 + g_2 + (1 - g_2)^2 + (1 - g_1)^2,$$

and is minimized under the constraints $g_1 \geq 0,\ g_2 \geq 0,\ g_1 + g_2 \leq \alpha$ for $g_1 = g_2 = \frac{1}{2}\alpha$ (and so $g_3 = 0$).

We thus obtain the following proposition, which gives a lower bound on the PoA that any Stackelberg strategy must satisfy for the example studied above, and hence for at least one instance with linear delay functions.

Proposition 5. *There is an instance with linear delay functions such that if* $(\boldsymbol{g}, \boldsymbol{x^g})$ *is an arbitrary Stackelberg strategy of weight* α *inducing a WE flow* $(\boldsymbol{h}, \boldsymbol{x^h})$ *for* $\mathcal{G}^{\boldsymbol{g}}$*, then*

$$PoA(\boldsymbol{g}) \geq \frac{4 - 2\alpha + \alpha^2}{3}.$$

Proof. For a given α, we know that the Stackelberg strategy minimizing $C(\boldsymbol{x^g} + \boldsymbol{x^h})$ is $\boldsymbol{g} = (\alpha/2, \alpha/2, 0)^T$ and induces the WE $\boldsymbol{h} = (0, 0, 1 - \alpha)^T$ for $\mathcal{G}^{\boldsymbol{g}}$. This flow has cost

$$C(\boldsymbol{x^g} + \boldsymbol{x^h}) = \frac{1}{2}\alpha + \frac{1}{2}\alpha + (1 - \frac{1}{2}\alpha)^2 + (1 - \frac{1}{2}\alpha)^2 = \alpha + 2(1 + \frac{\alpha^2}{4} - \alpha).$$

We obtain the lower bound of the proposition by dividing by the social optimum $C(\boldsymbol{x^o}) = \frac{3}{2}$, which is obtained for the flow $\boldsymbol{o}$ where half of the traffic is routed through A, half is routed through B, and no one takes the zigzag :

$$\frac{C(\boldsymbol{x^g} + \boldsymbol{x^h})}{C(\boldsymbol{x^o})} \geq \Big(\alpha + 2(1 + \frac{\alpha^2}{4} - \alpha)\Big)\frac{2}{3} = \frac{4 - 2\alpha + \alpha^2}{3}.$$

□

The bad news is that in general it is very hard to compute the optimal Stackelberg strategy. Below is a hardness result proved by Roughgarden, whose proof will be omitted.

Theorem 6. *For* $\alpha \in (0, 1)$*, the problem of computing the optimal Stackelberg strategy is NP-hard, even for instances in networks of parallel links with linear delay functions.*

But the good news is that the optimal Stackelberg strategy can be approximated by an efficient and simple algorithm. This has been known for a long time for single-commodity networks of parallel links, but until very recently no result was known for arbitrary network topologies. In a recent article, [Bonifaci, Harks & Schäfer, 2010], it was shown that a simple heuristic approach (the SCALE strategy) approaches the

for networks of arbitrary topology. The approach of these authors relies on the very intuitive SCALE strategy:

Definition 1 (SCALE Strategy)**.** The scale strategy consists in routing a fraction α of the traffic, by setting $\boldsymbol{g} = \alpha \boldsymbol{f^o}$, where $\boldsymbol{f^o}$ is a social optimum of $\mathcal{G}$.

We are now going to give the result of Bonifaci et. al. for linear delay functions.

Before giving the main theorem of this section, we need a technical definition, similar as the definition of $\beta(\mathcal{D})$ we used to study the PoA of continuous games: consider a game $\mathcal{G}$ with delay functions d_e in the family $\mathcal{D}$, let $(\boldsymbol{f^o}, \boldsymbol{x^o})$ be a social optimum and let λ be a positive scalar. For a Stackelberg strategy $(\boldsymbol{g}, \boldsymbol{x^g})$ inducing a Wardrop equilibrium $(\boldsymbol{h}, \boldsymbol{x^h})$ of $\mathcal{G}^{\boldsymbol{g}}$, we define

$$\omega_\lambda(\boldsymbol{g}, \mathcal{D}) := \sup_{\boldsymbol{d} \in \mathcal{D}} \max_{e \in E} \frac{x_e^o}{x_e^g + x_e^h} \frac{d_e(x_e^g + x_e^h) - \lambda d_e(x_e^o)}{d_e(x_e^g + x_e^h)}.$$

Proposition 7. *Let $\mathcal{G}$ be a NACG with delay functions d_e in the family $\mathcal{D}$, and let $(\boldsymbol{o}, \boldsymbol{x^o})$ be a social optimum. If $(\boldsymbol{g}, \boldsymbol{x^g})$ is a Stackelberg strategy satisfying $x_e^g \leq x_e^o$ for all $e \in E$, and if $\omega_\lambda(\boldsymbol{g}, \mathcal{D}) < 1$, then*

$$PoA(\boldsymbol{g}) \leq \frac{\lambda}{1 - \omega_\lambda(\boldsymbol{g}, \mathcal{D})}.$$

Note that this proposition virtually gives a bound for every scalar λ satisfying the technical condition $\omega_\lambda(\boldsymbol{g}, \mathcal{D}) < 1$.

Proof. Recall the variational characterization of a WE (Proposition 2 in Lecture #7): $(\boldsymbol{h}, \boldsymbol{x^h})$ is a WE of $\mathcal{G}^{\boldsymbol{g}}$ iff

$$\forall \boldsymbol{x'} \text{ feasible load for } \mathcal{G}^{\boldsymbol{g}}, \quad \sum_{e \in E} x_e^h \ d_e(x_e^g + x_e^h) \leq \sum_{e \in E} x'_e \ d_e(x_e^g + x_e^h).$$

In particular, for the flow $\boldsymbol{h'} = \boldsymbol{o} - \boldsymbol{g}$, which is feasible for $\mathcal{G}^{\boldsymbol{g}}$ and induces the load $\boldsymbol{x'} = \boldsymbol{x^o} - \boldsymbol{x^g}$, we obtain

$$\begin{aligned} & \sum_{e \in E} x_e^h \ d_e(x_e^g + x_e^h) \leq \sum_{e \in E} (x_e^o - x_e^g) \ d_e(x_e^g + x_e^h) \\ \iff \quad & C(\boldsymbol{x^g} + \boldsymbol{x^h}) \leq \sum_{e \in E} x_e^o d_e(x_e^g + x_e^h). \end{aligned}$$

Now, by using the definition of $\omega_\lambda(\boldsymbol{g}, \mathcal{D})$, we have

$$\forall e \in E, \quad \omega_\lambda(\boldsymbol{g}, \mathcal{D})(x_e^g + x_e^h) d_e(x_e^g + x_e^h) \geq x_e^o \big(d_e(x_e^g + x_e^h) - \lambda d_e(x_e^o) \big).$$

By combining the last two inequalities, we obtain:

$$C(\boldsymbol{x^g} + \boldsymbol{x^h}) \leq \sum_{e \in E} \omega_\lambda(\boldsymbol{g}, \mathcal{D})(x_e^g + x_e^h) d_e(x_e^g + x_e^h) + \lambda x_e^o d_e(x_e^o) = \omega_\lambda(\boldsymbol{g}, \mathcal{D}) C(\boldsymbol{x^g} + \boldsymbol{x^h}) + \lambda C(\boldsymbol{x^o})$$

If the condition $\omega_\lambda(\boldsymbol{g}, \mathcal{D}) < 1$ is fulfilled, this implies

$$C(\boldsymbol{x^g} + \boldsymbol{x^h}) \leq \frac{\lambda}{1 - \omega_\lambda(\boldsymbol{g}, \mathcal{D})} C(\boldsymbol{x^o}).$$

□

We can finally prove a theorem for the SCALE strategy and linear delays (or even, any function in the class $\mathcal{L}$ of delay functions d (continuous, nondecreasing) satisfying $\forall x \geq 0, \forall c \in [0,1], \ d(cx) \geq c d(x)$. We have already noted in an exercise that $\mathcal{L}$ actually contains all concave delay functions.

Theorem 8. *Let $\mathcal{G}$ be a game with delay functions in $\mathcal{L}$, $(\boldsymbol{o}, \boldsymbol{x^o})$ be a social optimum and $\boldsymbol{g} = \alpha \boldsymbol{o}$ be a SCALE strategy routing a fraction α of the traffic. Then,*

$$PoA(\boldsymbol{g}) \leq \frac{(1 + \sqrt{1 - \alpha})^2}{2(1 + \sqrt{1 - \alpha}) - 1}.$$

Proof. Let $\lambda \in (0,1)$. We first prove that $\omega_\lambda(\boldsymbol{g}, \mathcal{L}) \leq \max\{\frac{1-\lambda}{\alpha}, \frac{1}{4\lambda}\}$. To do this, define

$$\omega_e = \frac{x_e^o}{\alpha x_e^o + x_e^h} \frac{d_e(\alpha x_e^o + x_e^h) - \lambda d_e(x_e^o)}{d_e(\alpha x_e^o + x_e^h)}.$$

We consider two cases:

- If $\alpha x_e^o + x_e^h \geq x_e^o$, we define $\mu = \frac{x_e^o}{\alpha x_e^o + x_e^h} \leq 1$. Then, using that $d_e \in \mathcal{L}$ we obtain

$$\omega_e \leq \mu \frac{d_e(\alpha x_e^o + x_e^h) - \lambda d_e\big(\mu(\alpha x_e^o + x_e^h)\big)}{d_e(\alpha x_e^o + x_e^h)} \leq \mu(1-\lambda\mu) \leq \frac{1}{4\lambda},$$

  where the last inequality is obtained by taking the max of $\mu(1-\lambda\mu)$ over $\mu \in [0,1]$.

- otherwise ($\alpha x_e^o + x_e^h < x_e^o$), we have

$$\omega_e \leq \frac{x_e^o}{\alpha x_e^o + x_e^h} \frac{d_e(\alpha x_e^o + x_e^h) - \lambda d_e(\alpha x_e^o + x_e^h)}{d_e(\alpha x_e^o + x_e^h)} \leq \sup_{z \geq 0} \frac{x_e^o}{\alpha x_e^o + z}(1-\lambda) = \frac{1-\lambda}{\alpha},$$

  where for the first inequality we have used that d_e is nondecreasing.

These two cases show that $\omega_\lambda(\boldsymbol{g}, \mathcal{L}) \leq \max\{\frac{1-\lambda}{\alpha}, \frac{1}{4\lambda}\}$.

Now, we take the particular value $\lambda = \frac{1}{2}(1+\sqrt{1-\alpha})$, which is a value ≤ 1 for which the two arguments of the maximum in the upper bound of $\omega_\lambda(\boldsymbol{g}, \mathcal{L})$ coincide:

$$\frac{1-\lambda}{\alpha} = \frac{1}{4\lambda} = \frac{1}{2(1+\sqrt{1-\alpha})}.$$

So we have $\omega_\lambda(\boldsymbol{g}, \mathcal{L}) \leq \frac{1}{2(1+\sqrt{1-\alpha})} < 1$, and we can apply Proposition 7:

$$PoA \leq \frac{\lambda}{1-\omega_\lambda(\boldsymbol{g}, \mathcal{D})} = \frac{\lambda}{1-\frac{1}{4\lambda}} = \frac{4\lambda^2}{4\lambda - 1} = \frac{(1+\sqrt{1-\alpha})^2}{2(1+\sqrt{1-\alpha})-1}.$$

□

Observe that Theorem 8 shows that the SCALE strategy approximates the optimal Stackelberg strategy in the following sense: it provides a solution $\boldsymbol{g}$ such that $PoA(\boldsymbol{g}) \leq \frac{(1+\sqrt{1-\alpha})^2}{2(1+\sqrt{1-\alpha})-1}$, while we know from Proposition 5 that for some instance, the optimal Stackelberg strategy $\boldsymbol{g}^*$ satisfies $PoA(\boldsymbol{g}^*) \geq \frac{4-2\alpha+\alpha^2}{3}$. Those bounds are displayed in the figure at the bottom of this page. For all α, the upper bound is less than 1.12 times the lower bound. So the SCALE strategy is within 12% of the best possible bound we could have hoped to prove for any Stackelberg strategy.

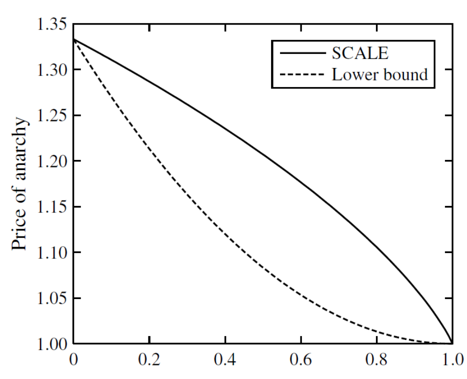


Figure 1: Upper bound for the PoA of the SCALE strategy (for all instances), vs. Lower bound for the PoA of *any* strategy (attained for some instances). Figure from [Bonifaci, Harks & Schäfer, 2010].