Lecture #1 Notes Summary

- Introduction of games in extensive form (tree) and normal form (matrix).
- Notions of Nash equilibrium, best responses, and dominance.

Administration

- Grade ? 50% of the exercise points are required to be qualified to pass the exam.
- Timing: Wed, Thu
- Language: English
- E-mails
- Knowledge in Linear Programming ?
- Website:
  `http://www.zib.de/sagnol/vorlesungen/Vorlesung_game_theory_transportation_networks_WS1314.html`

Introductive example

- Extensive form of a game: Tree games
- Necessity to define numerical payoffs
- If players are rational: reasonning by backward induction
- Extensive games can be converted to normal form games
- Never play Dominated strategies
- Nash equilibrium

Games in Extensive Form

A game in extensive form with N player $P_1, \dots, P_N$ is described by a game tree T, that is, a rooted tree with each non-terminal node labelled (*owned*) by a player P_i, and each leaf associated with a N-tuple of payoffs.

> **Definition 1** (Choice function)**.** Let T be a game tree. A *choice function* c for player P is a function that maps each node of T owned by P to one of his children $c(u) = v \in \mathrm{Ch}(u)$ (where $\mathrm{Ch}(u)$ denotes the set of child nodes of u).

But with a choice function we define actions at nodes that will never be reached. So we define

> **Definition 2** (Strategy)**.** A subtree S of T is said to be a *strategy* for player P, or a *choice subtree* for P, if the following holds:
>
> (i) If u is a node of S owned by P, then exactly one of the children of u belongs to S;
>
> (ii) If u is a node of S owned by another player, then all the children of u are in S;

Definition 3 (Game in extensive form)**.** A game in extensive form is defined by a triple $\Gamma = \Big(T, \{P_1, \ldots, P_N\}, \{\Sigma_1, \ldots, \Sigma_N\}\Big)$, where T is a game tree, $P_1, \ldots, P_N$ are the players of the game and Σ_i is the set of available strategies for player i

A game is *of perfect information* if only one player moves at a time and if each player knows every action of the players that moved before him at every point. A way to formalize this definition is:

Definition 4 (Perfect information)**.** A game $\Gamma = \Big(T, \{P_1, \ldots, P_N\}, \{\Sigma_1, \ldots, \Sigma_N\}\Big)$ is of *perfect information* if for every player P_i, Σ_i is the set of all possible choice subtrees for P_i in T.

The next proposition shows that the outcome of the game is well defined if we know the strategy S_i played by each player P_i.

Proposition 1. *Let $\Gamma = \Big(T, \{P_1, \ldots, P_N\}, \{\Sigma_1, \ldots, \Sigma_N\}\Big)$ be a game in extensive form. If each player P_i chooses a choice subtree $S_i \in \Sigma_i$, then the intersection of the S_i's form a path from the root of T to a terminal node z. We denote by $\pi_i(S_1, \ldots, S_N)$ the i^{th} component of the $N-$tuple payoff associated with that vertex z.*

Proof. cf. Exercise 2 of the worksheet #1. □

The payoffs can be used to define an essential "solution concept" of game theory: the concept of Nash equilibrium. Roughly speaking, it is a situation from which no player wants to depart if the others keep playing the same strategy.

Definition 5 (Nash equilibrium)**.** A N-tuple of strategies $S_1^*, \ldots, S_N^* \in \Sigma_1 \times \ldots \times \Sigma_N$ is a Nash equilibrium of the game iff for every i and for every strategy $S_i \in \Sigma_i$,

$$\pi_i(S_1^*, \ldots, S_i^*, \ldots, S_N^*) \geq \pi_i(S_1^*, \ldots, S_i, \ldots, S_N^*).$$

A crucial result from von Neumann and Morgenstern (1944) states the existence of a Nash equilibrium for a game in extensive form with perfect information:

Theorem 2. *Let $\Gamma = \Big(T, \{P_1, \ldots, P_N\}, \{\Sigma_1, \ldots, \Sigma_N\}\Big)$ be a game in extensive form. If Γ is of perfect information, then Γ has at least one Nash equilibrium.*

Proof. Proof by backward induction in the exercise 7 of the worksheet #1. □

Games in normal form

Every game $\Gamma = \Big(T, \{P_1, \ldots, P_N\}, \{\Sigma_1, \ldots, \Sigma_N\}\Big)$ can be converted to a *game in normal form*, i.e. a game defined by its payoff function only:

$$\boldsymbol{\pi} : \Sigma_1 \times \ldots \times \Sigma_N \mapsto \mathbb{R}^N, \qquad \boldsymbol{\pi}(S_1, \ldots, S_N) = \big[\pi_1(S_1, \ldots, S_N), \ldots, \pi_N(S_1, \ldots, S_N)\big]^T$$

Most often, we will study games in their normal form directly, without even giving their extensive form.

Definition 6 (Game in normal form)**.** Let $X_1, \ldots, X_N$ be finite sets and let $\boldsymbol{\pi}$ be a function that maps the cartesian product $X_1 \times \ldots \times X_N$ to $\mathbb{R}^N$. Then, $\boldsymbol{\pi}$ is called a *game in normal form* with the sets of *pure strategies* $X_1, \ldots, X_N$.

Not all the normal form games have a Nash equilibrium, as illustrated by the game of matching coins: P_1 and P_2 hold either a coin of 1$ or 5$ in their closed hand. They reveal their coin simultaneously: P_1 wins the coins if the coins are the same, while P_2 wins if the coins are different.

$$\begin{array}{cc} P_1/P_2 & \begin{array}{cc} \quad 1 \quad & \quad 5 \quad \end{array} \\ \begin{array}{c} 1 \\ 5 \end{array} & \begin{pmatrix} (1,-1) & (-1,1) \\ (-5,5) & (5,-5) \end{pmatrix} \end{array}$$

In order to cope with this problem, we introduce the concept of mixed strategy:

Definition 7 (Mixed strategy)**.** A mixed strategy for Player i is a probability distribution over $\Sigma_i = (S^i_1, \ldots, S^i_{n_i})$. It can be represented by a vector $\boldsymbol{p^i} \in \Delta_{n_i}$ of dimension n_i, where Δ_n stands for the probability simplex

$$\Delta_n := \left\{ \boldsymbol{p} \in \mathbb{R}^n : \forall k = 1, \ldots, n,\ p_k \geq 0, \qquad \sum_{k=1}^{n} p_k = 1. \right\}$$

The value p^i_k can be interpreted as the propability that Player P_i chooses Strategy S^i_k.

The payoff function $\boldsymbol{\pi}$ can be extended to mixed strategies as follows:

Definition 8 (Payoff for mixed strategies)**.** If every player P_i commits to a mixed strategy $\boldsymbol{p^i} \in \Delta_{n_i}$, the expected payoff of player P_i is

$$\tilde{\pi}_i(\boldsymbol{p^1}, \ldots, \boldsymbol{p^N}) = \sum_{(k_1,\ldots,k_N)} \left(\prod_{j=1}^{N} p^j_{k_j} \right) \pi_i(S^1_{k_1}, \ldots, S^N_{k_N}),$$

where the sum is carried over all the possible $N-$tuples of strategies. With a slight abuse of notation, we will still write π instead of $\tilde{\pi}$.

We next introduce the notion of *best responses*, which says what a player should play if the strategies of the others are fixed.

Definition 9. Denote by $\boldsymbol{p^{-i}} := (\boldsymbol{p^1}, \ldots, \boldsymbol{p^{i-1}}, \boldsymbol{p^{i+1}}, \ldots, \boldsymbol{p^N})$ the subprofile of mixed strategies played by the players other than P_i. We say that $\boldsymbol{p^i} \in \Delta_{n_i}$ is a *best response* to $\boldsymbol{p^{-i}}$ if

$$\pi_i(\boldsymbol{p^i}, \boldsymbol{p^{-i}}) = \max_{\boldsymbol{q} \in \Delta_{n_i}} \pi_i(\boldsymbol{q}, \boldsymbol{p^{-i}}).$$

The set of best responses for the player P_i to the subprofile of other's strategies $\boldsymbol{p^{-i}}$ is denoted by

$$BR_i(\boldsymbol{p^{-i}}) := \arg\max_{\boldsymbol{q} \in \Delta_{n_i}} \pi_i(\boldsymbol{q}, \boldsymbol{p^{-i}}).$$

Proposition 3. *Consider a game* $\boldsymbol{\pi}$ *with* N *players, and let* $\boldsymbol{p^{-i}} := (\boldsymbol{p^1}, \ldots, \boldsymbol{p^{i-1}}, \boldsymbol{p^{i+1}}, \ldots, \boldsymbol{p^N})$ *be the subprofile of mixed strategies played by the players other than* P_i.

(i) Player P_i *has always a pure strategy* S_k^i *that is a best response to* $\boldsymbol{p^{-i}}$.

(ii) If the pure strategies of P_i *in best response relationship to* $\boldsymbol{p^{-i}}$ *are* $(S_{k_1}^i, S_{k_2}^i, \ldots, S_{k_r}^i)$, *then the set of best responses* $BR_i(\boldsymbol{p^{-i}})$ *is the set of mixtures of the pure strategies* $(S_{k_1}^i, S_{k_2}^i, \ldots, S_{k_r}^i)$. *In other words,* $BR_i(\boldsymbol{p^{-i}})$ *is a nonempty convex polytope whose extreme points are the pure best response strategies to* $\boldsymbol{p^{-i}}$.

Proof. cf. Exercise 6 of the worksheet #1. □

The concept of Nash equilibrium can be translated to mixed strategies in a straightforward way, and we give an equivalent definition in terms of best responses:

Definition 10 (Nash equilibrium in mixed strategies)**.** Consider a game $\boldsymbol{\pi}$ with N players, where every player P_i has the choice between n_i pure strategies. A N-tuple of mixed strategies $\boldsymbol{p_*^1}, \ldots, \boldsymbol{p_*^N} \in \Delta_{n_1} \times \ldots \times \Delta_{n_N}$ is a Nash equilibrium of the game iff for every i and for every strategy $\boldsymbol{p_i} \in \Delta_{n_i}$,

$$\pi_i(\boldsymbol{p_*^1}, \ldots, \boldsymbol{p_*^i}, \ldots, \boldsymbol{p_*^N}) \geq \pi_i(\boldsymbol{p_*^1}, \ldots, \boldsymbol{p^i}, \ldots, \boldsymbol{p_*^N}).$$

In other words, for every player P_i the strategy $\boldsymbol{p_*^i}$ is a *best response* to the subprofile of other players' strategies:

$$\forall i = 1, \ldots, N, \qquad \boldsymbol{p_*^i} \in BR_i(\boldsymbol{p_*^{-i}})$$

A fundamental result of Nash (1951) is the following. We omit his proof, which is not constructive and based on the Brouwer fixed-point theorem.

Theorem 4 (Existence of a mixed Nash equilibrium)**.** *Let* $\boldsymbol{\pi}$ *be a N-player game in normal form. Then there exists at least one Nash equilibrium of mixed strategies.*

An important notion of game theory is the relation of dominance, which can be used to eliminate irrelevant pure strategies (and hence reduce the size of the game).

Definition 11 (Dominance)**.** Let $\boldsymbol{\pi}$ be a game given in normal form.

- A Strategy S_i^* of Player P_i *strongly dominates* S_i' iff for all strategies $S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_N$ of the other players,
$$\pi_i(S_1, \ldots, S_i^*, \ldots, S_N) > \pi_i(S_1, \ldots, S_i', \ldots, S_N).$$
- A Strategy S_i^* *weakly dominates* S_i' iff for all strategies $S_{-i} := \big(S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_N\big)$ of the other players,
$$\pi_i(S_i^*, S_{-i}) \geq \pi_i(S_i', S_{-i}),$$
and the inequality is strict for at least one subprofile S_{-i} of other players' strategies.
- A Strategy $S_i \in \Sigma_i$ is called strongly (resp. weakly) *dominant* if it strongly (resp. weakly) dominates all the other available strategies $S_i' \in \Sigma_i$.
- A Strategy $S_i \in \Sigma_i$ is called strongly (resp. weakly) *dominated* if there exists another strategy $S_i' \in \Sigma_i$ that strongly (resp. weakly) dominates it.

Proposition 5. *Consider a $N-$player game $\boldsymbol{\pi}$.*

(i) If player P_i has a strongly dominated strategy S^i_k, and $(\boldsymbol{p^1},\ldots,\boldsymbol{p^N})$ is a mixed Nash equilibrium, then $p^i_k=0$. In particular, S^i_k cannot be part of any pure Nash equilibrium.

(ii) If player P_i has a strongly dominant strategy S^i_k, and $(\boldsymbol{p^1},\ldots,\boldsymbol{p^N})$ is a mixed Nash equilibrium, then $p^i_k=1$. In particular, the pure strategy S^i_k must be part of every pure Nash equilibrium of the game.

(iii) If player P_i has a weakly dominated strategy S^i_k, then there exists a mixed Nash equilibrium $(\boldsymbol{p^1},\ldots,\boldsymbol{p^N})$ such that $p^i_k=0$.

Proof. (i) Assume without loss of generality that $i=1$, and that the strategy S^1_2 of player P_1 is strongly dominated by S^1_1. Let $(\boldsymbol{p^1},\ldots,\boldsymbol{p^N})$ be a (mixed) Nash equilibrium of the game. For all choice $[k]=(k_2,\ldots,k_N)$ of strategy indices for the other players $P_2,\ldots,P_N$, we define $S^{-1}_{[k]}=(S^2_{k_2},\ldots,S^N_{k_N})$ and $p_{[k]}=\prod_{i=2}^N p^i_{k_i}$.

With this notation, the expected payoff of player P_1 can be written as

$$\pi_1(\boldsymbol{p^1},\ldots,\boldsymbol{p^N})=\sum_{j=1}^{n_1} p^1_j \underbrace{\sum_{[k]} p_{[k]}\ \pi_1(S^1_j,S^{-1}_{[k]})}_{\alpha_j},$$

where the sum over j goes over the n_1 pure strategies of P_1 and the sum over $[k]$ goes over all possible $(N-1)$-tuples of strategies for the other players. Since S^1_2 is strongly dominated by S^1_1, for all $S^{-1}_{[k]}$ we have $\pi_1(S^1_2,S^{-1}_{[k]})<\pi_1(S^1_1,S^{-1}_{[k]})$. Clearly, $p_{[k]}$ must be positive for at least one index $[k]$, so we have $\alpha_2<\alpha_1$. Finally, we know that $\boldsymbol{p_1}$ is a best response to $(\boldsymbol{p^2},\ldots,\boldsymbol{p^N})$. This means that $\boldsymbol{p_1}$ maximizes $\sum_j p^1_j\alpha_j$ over Δ_{n_1}. So $\boldsymbol{p_1}$ cannot gives a positive weight to $\alpha_2<\alpha_1$, i.e. $p^1_2=0$.

(ii) If S^i_k is dominant, then the other strategies $S^i_1,\ldots,S^i_{k-1},S^i_{k+1},\ldots,S^i_{n_i}$ of player P_i are dominated. So if $(\boldsymbol{p^1},\ldots,\boldsymbol{p^N})$ be a (mixed) Nash equilibrium of the game, then by point (i) we know that $p^i_1=\ldots=p^i_{k-1}=p^i_{k+1}=\ldots=p^i_{n_i}=0$, which implies $p^i_k=1$.

(iii) cf. Exercise 4 of the worksheet #1.

□