Lecture #2 Notes Summary

- Zero-Sum matrix games: Saddle points, Minimax theorem
- Bimatrix games: Classical examples, graphical method to find mixed equilibria of $2 \times 2-$games

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A game in normal form with two players is called a bimatrix game. The reason is that player when P_1 has m strategies and player P_2 has n strategies, the payoff function $\boldsymbol{\pi}$ can be given by a pair of $m \times n$ matrices.

The game of matching coins presented in the first lecture

$$\begin{array}{ccc} P_1/P_2 & 1 & 5 \\ 1 & (1,-1) & (-1,1) \\ 5 & (-5,5) & (5,-5) \end{array}$$

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can be represented by a pair of matrices (A, B), where $A = \begin{pmatrix} 1 & -1 \\ -5 & 5 \end{pmatrix}$ contains the payoffs π_1 of the first player and $B = \begin{pmatrix} -1 & 1 \\ 5 & -5 \end{pmatrix}$ contains the payoffs of the second player.

Matrix Games (or Two-Player Zero-sum Games)

If $A + B = 0$ (as this is the case for the game of matchnig coins), the game is called a *zero-sum* game. So we only need to give the matrix A to define the payoff function $\boldsymbol{\pi}$ of this game.

Definition 1 (Matrix game)**.** A *matrix game* with the matrix A of size $m \times n$ is a zero-sum game with two players, such that:

- the player P_1 has m strategies: $\Sigma_1 = \{S_1^1, \ldots, S_m^1\}$
- the player P_2 has n strategies: $\Sigma_2 = \{S_1^2, \ldots, S_n^2\}$
- the payoff function is defined by $\pi_1(S_i^1, S_j^2) = A_{i,j}$ (and $\pi_2(S_i^1, S_j^2) = -A_{i,j}$).

We say that P_1 is the *row player* or the *max player* (she selects a row i of the matrix and wants to maximize $A_{i,j}$), while P_2 is the *column player* or the *min player* (she selects a column j of the matrix and wants to minimize $A_{i,j}$).

Proposition 1. *Let A be a matrix game. If player P_1 commits to the mixed strategy $\boldsymbol{p} \in \Delta_m$ and player P_2 commits to the mixed strategy $\boldsymbol{q} \in \Delta_n$, then the expected payoff of player P_1 is*

$$\pi_1(\boldsymbol{p}, \boldsymbol{q}) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j A_{i,j} = \boldsymbol{p}^T A \boldsymbol{q}.$$

Proof. Use the definition of expected payoffs and rewrite the expression with matrix notation. □

For matrix games, a pure Nash equilibrium is called a *saddle point* of a matrix.

Definition 2 (Saddle point)**.** Let A be a $m \times n-$matrix. An entry $A_{p,q}$ of A is a *saddle point* of A is $A_{p,q}$ is simultaneously a maximum in the column q and a minimum in the row p.

Saddle points of following matrices are boxed:

$$\begin{pmatrix} \boxed{1} & 2 \\ 0 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \boxed{7} & 8 & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & \boxed{2} & \boxed{2} \\ 2 & 1 & 1 \\ 3 & \boxed{2} & \boxed{2} \end{pmatrix}$$ #1

Theorem 2. *If $M_{i,j}$ and $M_{k,l}$ are saddle points of M, then $M_{i,l}$ and $M_{k,j}$ are also saddle points and*

$$M_{i,j} = M_{k,l} = M_{i,l} = M_{k,j}.$$

Proof. $M_{i,j}$ is a maximum of column j, so $M_{k,j} \leq M_{i,j}$, and it is also a minimum of row i, so $M_{i,j} \leq M_{i,l}$. $M_{k,l}$ is a maximum of column l, so $M_{i,l} \leq M_{k,l}$, and it is also a minimum of row k, so $M_{k,l} \leq M_{k,j}$. Thus, we have

$$M_{k,j} \leq M_{i,j} \leq M_{i,l} \leq M_{k,l} \leq M_{k,j},$$

which shows that these 4 values are equal. In particular, $M_{i,l} = M_{i,j}$ is a minimum in row i, and $M_{i,l} = M_{k,l}$ is a maximum in column l, so $M_{i,l}$ is a saddle point. Similarly we can see that $M_{k,j}$ is a saddle point. □

Definition 3 (maximin values)**.** The *maximin* and *minimax values* of M are defined respectively as

$$\mu_r(M) = \max_i \min_j M_{i,j}$$

$$\mu_c(M) = \min_j \max_j M_{i,j}$$

A strategy S_i^1 of the row player is called *maximin pure strategy* if $\min_j M_{i,j} = \mu_r(M)$.
A strategy S_j^2 of the column player is called *minimax pure strategy* if $\max_i M_{i,j} = \mu_c(M)$.

Theorem 3. *For any matrix M,*

$$\mu_r(M) \leq \mu_c(M).$$

Moreover $\mu_r(M) = \mu_c(M)$ if and only if M has a saddle point.

Proof. For every $l \in \{1, \ldots, n\}$, we have $\mu_r(M) = \max_i \min_j M_{i,j} \leq \max_i M_{i,l}$. Hence,

$$\mu_r(M) \leq \min_l \max_i M_{i,l} = \mu_c(M).$$

Now, let $M_{p,q}$ be a saddle point. We have $\max_i M_{i,q} = M_{p,q}$ and so $\mu_c(M) \leq M_{p,q}$. Similarly, $\min_j M_{p,j} = M_{p,q}$ implies $\mu_r(M) \geq M_{p,q}$. So we have $\mu_c(M) \leq \mu_r(M)$, which proves the first side of the equivalence.

Conversely, assume that $\mu_c(M) = \mu_r(M)$. Choose a maximin pure strategy with the index p and a minimax pure strategy with the index q. We have $\mu_r(M) = \min_j M_{p,j}$, and let l be an index such that $M_{p,l} = \mu_r(M) = \mu_c(M)$. Since the column q is minimax, we have $\mu_c(M) = \max_i M_{i,q}$. Thus $M_{p,l} = \max_i M_{i,q} \geq M_{p,q}$, but l has be chosen so that $M_{p,l}$ is a minimum in its row, so $M_{p,l} = M_{p,q}$ and $M_{p,q}$ is also a minimum in its row. Finally,

$$M_{p,q} = M_{p,l} = \max_i M_{i,q}$$

is a maximum in its column, and so $M_{p,q}$ is a saddle point. □

We are now going to define the counterpart of maximin values for mixed strategies:

Definition 4 (row and column values)**.** The *row* and *column values* of a matrix $M \in \mathbb{R}^{m\times n}$ are defined respectively as

$$v_r(M) = \max_{\boldsymbol{p}\in\Delta_m} \min_{\boldsymbol{q}\in\Delta_n} \boldsymbol{p}^T M \boldsymbol{q} = \max_{\boldsymbol{p}\in\Delta_m} \min_{j} \left(\boldsymbol{p}^T M\right)_j$$

$$v_c(M) = \min_{\boldsymbol{q}\in\Delta_n} \max_{\boldsymbol{p}\in\Delta_m} \boldsymbol{p}^T M \boldsymbol{q} = \min_{\boldsymbol{q}\in\Delta_n} \max_{i} \left(M\boldsymbol{q}\right)_i$$

A mixed strategy $\boldsymbol{p}$ of the row player is called *optimal* if $\min_j \left(\boldsymbol{p}^T M\right)_j = v_r(M)$.
A mixed strategy $\boldsymbol{q}$ of the column player is called *optimal* if $\max_i \left(M\boldsymbol{q}\right)_i = v_c(M)$.

Theorem 4. *Let $\bar{\boldsymbol{p}}$ and $\bar{\boldsymbol{q}}$ be optimal strategies of the row and column players for a matrix game M. Then,*

$$\mu_r(M) \leq v_r(M) \leq \bar{\boldsymbol{p}}^T M \bar{\boldsymbol{q}} \leq v_c(M) \leq \mu_c(M).$$

Proof. cf. Exercise 2 of the Worksheet #2. □

Theorem 5 (minimax theorem)**.** *Let M be a $m\times n$ matrix game. Then the row and the column players have optimal mixed strategies $\bar{\boldsymbol{p}}$ and $\bar{\boldsymbol{q}}$, and the row and column values of the game are equal:*

$$v_r(M) = \bar{\boldsymbol{p}}^T M \bar{\boldsymbol{q}} = v_c(M).$$

The common row and column values define the value of the game*, which we denote by $v(M)$.*

Proof. We know from the existence theorem of Nash (cf. first Lecture) that a Nash equilibrium $(\boldsymbol{p}^*, \boldsymbol{q}^*)$ exists. So $\boldsymbol{q}^*$ is a best response to $\boldsymbol{p}^*$, i.e. ${\boldsymbol{p}^*}^T M \boldsymbol{q}^* = \min_{q\in\Delta_n} {\boldsymbol{p}^*}^T M \boldsymbol{q}$. Combining this equality with

$$\min_{q\in\Delta_n} {\boldsymbol{p}^*}^T M \boldsymbol{q} \leq \max_{p\in\Delta_m} \min_{q\in\Delta_n} \boldsymbol{p}^T M \boldsymbol{q} = v_r(M)$$

we obtain ${\boldsymbol{p}^*}^T M \boldsymbol{q}^* \leq v_r(M)$. Similarly, $\boldsymbol{p}^*$ is a best response to $\boldsymbol{q}^*$ yields ${\boldsymbol{p}^*}^T M \boldsymbol{q}^* \geq v_c(M)$.

Now, note that optimal strategies $\bar{\boldsymbol{p}}$ and $\bar{\boldsymbol{q}}$ for the row and column player exist (optimization of a continuous function over a compact set). So we can apply Theorem 4, and we obtain:

$$v_r(M) \leq \bar{\boldsymbol{p}}^T M \bar{\boldsymbol{q}} \leq v_c(M) \leq {\boldsymbol{p}^*}^T M \boldsymbol{q}^* \leq v_r(M),$$

which proves that all these values are equal. □

Corollary 6. *Let M be a $m\times n$ matrix game. The following statements are equivalent.*

(i) $(\boldsymbol{p}^*, \boldsymbol{q}^*)$ *is a pair of optimal strategies;*

(ii) $(\boldsymbol{p}^*, \boldsymbol{q}^*)$ *is a Nash equilibrium.*

Proof. $(i) \Rightarrow (ii)$: If $(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}})$ are optimal, then $\min_{\boldsymbol{q}\in\Delta_n} \bar{\boldsymbol{p}}^T M \boldsymbol{q} = v_r(M) = \bar{\boldsymbol{p}}^T M \bar{\boldsymbol{q}}$, so $\bar{\boldsymbol{q}}$ is a minimizer of $\bar{\boldsymbol{p}}^T M \boldsymbol{q}$ over Δ_n, i.e. $\bar{\boldsymbol{q}} \in BR_2(\bar{\boldsymbol{p}})$. Similarly, $\max_{\boldsymbol{p}\in\Delta_m} \boldsymbol{p}^T M \bar{\boldsymbol{q}} = v_c(M) = \bar{\boldsymbol{p}}^T M \bar{\boldsymbol{q}}$ tells us that $\bar{\boldsymbol{p}} \in BR_1(\bar{\boldsymbol{q}})$.

$(ii) \Rightarrow (i)$: If $(\boldsymbol{p}^*, \boldsymbol{q}^*)$ is a Nash equilibrium, then $\boldsymbol{p}^* \in BR_1(\boldsymbol{q}^*)$ implies

$${\boldsymbol{p}^*}^T M \boldsymbol{q}^* = \max_{\boldsymbol{p}\in\Delta_m} \boldsymbol{p}^T M \boldsymbol{q}^*,$$

and we know that ${\boldsymbol{p}^*}^T M \boldsymbol{q}^* = v(M) = v_c(M)$. So $\boldsymbol{q}^*$ is an optimal strategy of the column player. Similarly, $\boldsymbol{q}^* \in BR_2(\boldsymbol{p}^*)$ implies $v_r(M) = v(M) = {\boldsymbol{p}^*}^T M \boldsymbol{q}^* = \min_{\boldsymbol{q}\in\Delta_n} {\boldsymbol{p}^*}^T M \boldsymbol{q}$, which tells that $\boldsymbol{p}^*$ is an optimal strategy for the row player. □

We also have a result which is the counterpart of Theorem 2 for mixed strategies:

Corollary 7. *If $\boldsymbol{p}$ and $\boldsymbol{r}$ are both optimal strategies of the row player, and $\boldsymbol{q}$ and $\boldsymbol{s}$ are both optimal for the column player, then*

$$\boldsymbol{p}^T M \boldsymbol{q} = \boldsymbol{p}^T M \boldsymbol{s} = \boldsymbol{r}^T M \boldsymbol{q} = \boldsymbol{r}^T M \boldsymbol{s}.$$

Proof. Clear from the minimax theorem, and all these payoffs must be equal to $v(M)$. □

We next define symmetric games, which are games in which both players are indistinguishable.

Definition 5 (symmetric game)**.** A matrix game is called symmetric if the matrix M of the game is skew-symmetric ($M = -M^T$).

Rock Paper Scissors

P_1 / P_2	R	P	S
R	0	−1	1
P	1	0	−1
S	−1	1	0

#2

Theorem 8. *The value of a symmetric game is zero. Moreover if $\boldsymbol{p}$ is optimal for the first player, then $\boldsymbol{p}$ is also an optimal strategy for the column player*

Proof. Let $\boldsymbol{p}$ be optimal for the row player, i.e. $\boldsymbol{p}$ maximizes

$$\min_j (\boldsymbol{p}^T M)_j = \min_j (M^T \boldsymbol{p})_j = \min_j -(M\boldsymbol{p})_j = -\max_j (M\boldsymbol{p})_j,$$

so $\boldsymbol{p}$ minimizes $\max_j (M\boldsymbol{p})_j$, i.e. $\boldsymbol{p}$ is optimal for the column player.

The value of the game is thus $\boldsymbol{p}^T M \boldsymbol{p} = \sum_{i,j} p_i p_j M_{i,j} = \sum_{i<j} p_i p_j M_{i,j} + \underbrace{\sum_{j<i} p_i p_j M_{i,j}}_{=-\sum_{j<i} p_i p_j M_{j,i}} = 0.$ □

Bimatrix Games (or Two-Player Non-Zero-Sum Games)

In non-zero-sum games, the situation is not so clear anymore. We will see that Nash equilibria might lead to situations that are *bad* for both players. Moreover, using Nash equilibria to predict the outcome of the game is perilous, since several equilibria might coexist and the problem of selecting a Nash equilibrium is challenging.

Prisoner's Dilemma:

P_1/P_2	plead guilty	accuse the other one
plead guilty	$(-2,-2)$	$(-10,-1)$
accuse the other one	$(-1,-10)$	$\boxed{(-5,-5)}$

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Chicken Game: #2

P_1/P_2	turn	don't turn
turn	$(0,0)$	$\boxed{(-2,5)}$
don't turn	$\boxed{(5,-2)}$	$(-10,-10)$

Battle of the sex: #3

♂/ ♀	Football	Ballet
Football	$\boxed{(10,3)}$	$(2,2)$
Ballet	$(0,0)$	$\boxed{(3,10)}$

Graphical method to find the Nash equilibria of $2\times 2-$ bimatrix games

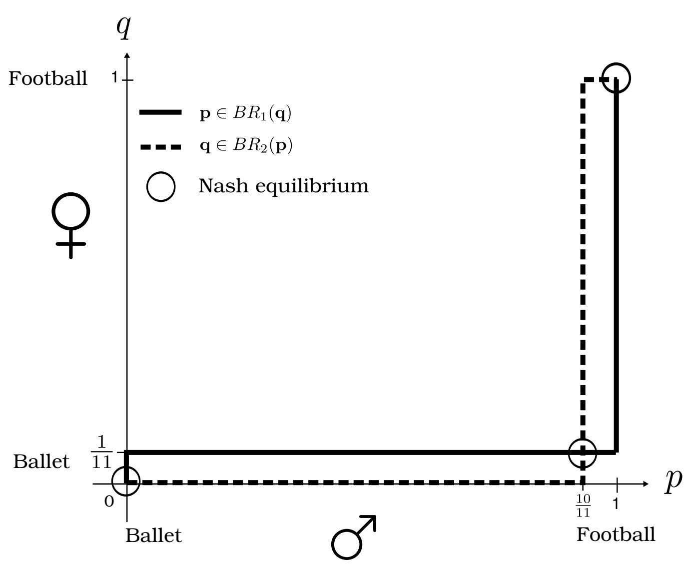
Define the sets

$$S_1 = \{(\boldsymbol{p},\boldsymbol{q}) \in \Delta_m \times \Delta_n : \quad \boldsymbol{p} \in BR_1(\boldsymbol{q})\}$$

$$S_2 = \{(\boldsymbol{p},\boldsymbol{q}) \in \Delta_m \times \Delta_n : \quad \boldsymbol{q} \in BR_2(\boldsymbol{p})\}$$

By definition, the set of Nash equilibria is $S_1 \cap S_2$. For $2\times 2-$ bimatrix games, these sets can be plotted in the square $[0,1]\times[0,1]$.

Consider the game of Battle of Sex (Example #3). We write $\boldsymbol{p} = [p, 1-p]^T \in \Delta_2$ and $\boldsymbol{q} = [q, 1-q]^T \in \Delta_2$. For the male player, the strategy *Football* is a best response to $\boldsymbol{q}$ iff $10q + 2(1-q) \geq 3(1-q) \Leftrightarrow q \geq \frac{1}{11}$. Similarly *Ballet* is a best response to $\boldsymbol{q}$ iff $q \leq \frac{1}{11}$.
Now for the female player, the strategy *Football* is a best response to $\boldsymbol{p}$ iff $3p \geq 2p + 10(1-p) \Leftrightarrow p \geq \frac{10}{11}$. Similarly *Ballet* is a best response to $\boldsymbol{p}$ iff $p \leq \frac{10}{11}$.
We know that the set of best responses are the convex hull of the pure best responses. So if $q \leq \frac{1}{11}$, every $\boldsymbol{p} \in \Delta_2$ is a best response to $\boldsymbol{q}$. Similarly $BR_2\big([\frac{10}{11}, \frac{1}{11}]^T\big) = \Delta_2$. The best responses sets are displayed on the following plot: #4



This shows that this bi-matrix game has two pure equilibria (Football, Football), (Ballet, Ballet) and one mixed equilibrium $\big([\frac{10}{11}, \frac{1}{11}]^T,\ [\frac{1}{11}, \frac{10}{11}]^T\big)$.