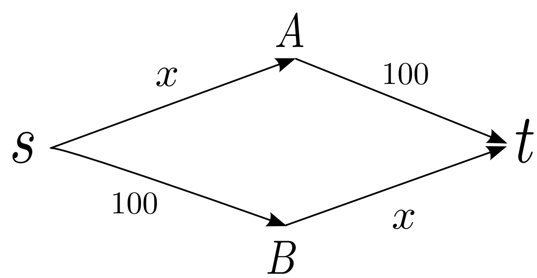
Lecture #6 Notes Summary

Braess paradox, atomic and non-atomic congestion games.

Introduction to congestion games: Braess' paradox

Consider the following Graph:



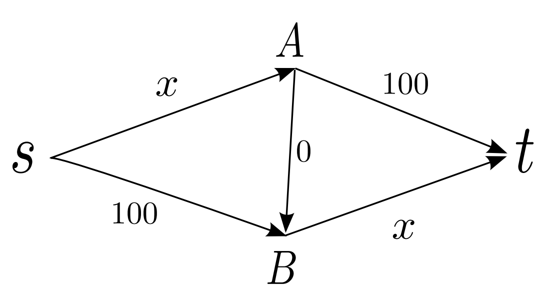
100 drivers want to drive from s to t.
Each driver has two strategies ($s \to A \to t$ and $s \to B \to t$).
The edges $s \to B$ and $A \to t$ always take 100 minutes to cross, regardless of the number of drivers taking those edges.
The edges $s \to A$ and $B \to t$ take x minutes to cross if x drivers use those edges.

It seems to be natural to predict that the drivers will split equally over the 2 paths, so the journey time of every driver will be $100 + 50 = 150$ minutes. This actually corresponds to a Nash equilibrium of this game. Indeed, Assume that the drivers are split 50-50 over the 2 paths. If a single driver changes his strategy, his journey time will increase from 150 to 151 minutes.
Note that this routing game can be seen as a standard N-player game (with $N = 100$), cf. Lecture #1. If the strategies chosen by the 100 players are $S^1, \ldots, S^{100} \in \{A, B\}^{100}$, we can write the payoff function of the game in normal form as:

$$\pi_i(S^1, \ldots, S^{100}) = 100 + \#\{j \in [100] : \; S^j = S^i\}$$

where $\#S$ denotes the cardinal of S.

Now, assume that a new high-speed motorway is built from A to B. This road is so fast that we attach a delay of 0 minute to $A \to B$.



Now the 50-50 solution is not a Nash Equilibrium anymore. Because in this situation every driver has an incentive to take the new route $s \to A \to B \to t$ (the journey time decreases to 101 minutes).

In fact, we can continue this reasonning until all the drivers take the new route. This situation is a Nash equilibrium, because if all the drivers take $s \to A \to B \to t$, no driver has a (strict) incentive to switch to $s \to A \to t$ or $s \to B \to t$ (the journey time would remain 200 minutes).

This fact is known as the *Braess' Paradox*: adding an edge to the network did not improve the life of the players. On the contrary, it generated a worse situation (journey time of 200 minutes vs. 150 minutes)!

In congestion games, we mainly study three questions:

- How do we compute a Nash Equilibrium
- Is it reasonnable to assume that drivers will behave according to a NE ?
- How bad can Braess' Paradox be ?

Atomic Congestion Games

Definition 1 (Atomic Congestion Game)**.** An *atomic congestion game* $(E, \boldsymbol{S}, \boldsymbol{w}, \boldsymbol{d})$ is a $N-$player game defined by

- A set E of congestible elements.
- A strategy space $\boldsymbol{S} = S^i \times \ldots \times S^N$ and a weight vector $\boldsymbol{w} \in \mathbb{R}_+^N$: each player $i \in [N]$ has a nonnegative weight w_i and a finite set of strategies S^i, such that every strategy $P \in S^i$ is a subset of E ($P \subseteq E$).
- For each element $e \in E$, a delay function $d_e : \mathbb{R}_+ \to \mathbb{R}_+$.

Given a strategy $P_i \in S^i$ for each player i, we define the *load* of element e:

$$x_e := \sum_{\{i \in [N]:\ e \in P_i\}} w_i,$$

and the payoff of player i is $\pi_i(P_1, \ldots, P_N) := -\sum_{e \in P_i} d_e(x_e)$.

Proposition 1. *A strategy profile* $(P_1, \ldots, P_N) \in \boldsymbol{S}$ *is a Nash Equilibrium of an atomic congestion game* $(E, \boldsymbol{S}, \boldsymbol{w}, \boldsymbol{d})$ *if and only if*

$$\forall i \in [N],\ \forall Q_i \in S^i,\ \sum_{e \in P_i} d_e(x_e) \leq \sum_{e \in P_i \cap Q_i} d_e(x_e) + \sum_{e \in Q_i \setminus P_i} d_e(x_e + w_i).$$

Proof. Let $P_1, \ldots, P_N$ be a strategy profile and let $\boldsymbol{x}$ be the associated vector of loads. Denote by $\boldsymbol{x'}$ the load vector obtained if player i swaps unilaterally her strategy from P_i to Q_i. By definition,

$$x'_e = \begin{cases} x_e & \text{if } e \in P_i \cap Q_i \\ x_e + w_i & \text{if } e \in Q_i \setminus P_i \\ x_e - w_i & \text{if } e \in P_i \setminus Q_i \end{cases}$$

We show that the $P_i \in BR_i(P_{-i})$ iff the condition of the theorem holds:

$$P_i \in BR_i(P_{-i}) \Longleftrightarrow \forall Q_i \in S^i,\ \sum_{e \in P_i} d_e(x_e) \leq \sum_{e \in Q_i} d_e(x'_e) = \sum_{e \in P_i \cap Q_i} d_e(x_e) + \sum_{e \in Q_i \setminus P_i} d_e(x_e + w_i).$$

□

Note that we are only considering *Pure* Nash Equilibria. So there is no guarantee of existence of an equilibrium. An example without equilibrium will be seen in Exercise 4 of Worksheet #6. We will now show that a Nash equilibrium always exists in unweighted atomic congestion games, i.e. when $w_i = 1$ for all $i \in [N]$. Moreover, the best response dynamics always converge to a Nash equilibrium, so it seems natural to assume that rational users will find it. To do this, we define the potential function of the game:

$$\Phi(P_1, \ldots, P_N) = \sum_{e \in E} \sum_{k=0}^{x_e} d_e(k).$$

Note that for all e the sum $\sum_{k=1}^{x_e} d_e(k)$ can be thought as a *discrete integral* of d_e.

Theorem 2. *In unweighted atomic congestion games ($w_i = 1$ for all $i \in [N]$),*

- *Every minimum of Φ is a Nash equilibrium;*
- *Iterative best responses find a Nash equilibrium.*

Proof. There is a finite number of players, and each player has a finite number of available strategies, so Φ admits a global minimum $(P_1, \ldots, P_N)$. We claim that this profile is a Nash equilibrium of the game. Assume the contrary. Then, there exists a player i and a strategy $Q_i \in S^i$ such that

$$\sum_{e \in P_i \cap Q_i} d_e(x_e) + \sum_{e \in Q_i \setminus P_i} d_e(x_e + 1) - \sum_{e \in P_i} d_e(x_e) < 0. \tag{1}$$

(remember that $w_i = 1$). Switching from P_i to Q_i has the following effect on the potential Φ: for the edges $e \in Q_i \setminus P_i$ there is new term in the sum, $d_e(x_e + 1)$, and for the edges $e \in P_i \setminus Q_i$ the sum looses the term $d_e(x_e)$. So we have

$$\Phi(P_1, \ldots, Q_i, \ldots, P_N) - \Phi(P_1, \ldots, P_i, \ldots, P_N) = \sum_{e \in Q_i \setminus P_i} d_e(x_e + 1) - \sum_{e \in P_i \setminus Q_i} d_e(x_e).$$

It is easy to see that this expression is the same as the left hand side of (**??**), so it must be negative, which contradicts the optimality of $(P_1, \ldots, P_N)$.

Next, observe that when a player switches from strategy P_i to strategy Q_i, the decrease in the total delay for this player is equal to the change in the potential function Φ. Thus, if players keep changing their strategies for a better one, this process (called *iterated best responses*, or *best responses dynamics*) will end up in a mininum of Φ, and hence a Nash equilibrium. □

There is another situation where the existence of a Nash Equilibrium can be stated:

Theorem 3. *Consider an atomic congestion game $(E, \boldsymbol{S}, \boldsymbol{w}, \boldsymbol{d})$ where all the delay functions are affine ($\forall e \in E$, $d_e(x) = a_e x + b_e$ for some $a_e, b_e \in \mathbb{R}_+$). Then, iterated best responses find a Nash Equilibrium.*

Proof. The proof makes use of another potential function:

$$\Phi(P_1, \ldots, P_N) = \sum_{e \in E} \left(x_e d_e(x_e) + \sum_{\{i \in [N]:\ e \in P_i\}} w_i d_e(w_i) \right),$$

cf. Exercise 2 of the worksheet #6. □

Non-atomic Congestion Games

We define the class of *non-atomic* congestion games. A game of this class can be thought as the limit of an atomic game, when the number of players sharing a set of strategies $S = \{P_1, \dots, P_k\}$ goes to infinity. The *players* of a nonatomic game are infinitesimally small; thus, we rather define the game with respect to N *types* of players. Typically a *type* regroup players that belong to the same origin-destination pair (s,t), and we are interested in the distribution of the traffic over the different $(s,t)-$paths.

Definition 2 (Non-Atomic Congestion Games)**.** An *non-atomic congestion game* $(E, \mathcal{S}, \boldsymbol{w}, \boldsymbol{d})$ is defined by

- A set E of congestible elements.
- A disjoint union $\mathcal{S} = \biguplus_{i\in[N]} S_i$ of strategy spaces and a weight vector $\boldsymbol{w} \in \mathbb{R}_+^N$. Instead of N players, there are now N types of players, and w_i can be intepreted as the quantity of players of type i.
- The players of type i can be split arbitrarily over the strategies $P \in S^i$. Formally, there is a flow $\boldsymbol{f} \in (\mathbb{R}_+)^{\mathcal{S}}$ such that f_P represents the amount of players choosing the strategy P and

  $$\forall i \in [N], \sum_{P\in S^i} f_P = w_i. \tag{2}$$

  A nonnegative flow $\boldsymbol{f}$ satisfying Eq. (**??**) is called *feasible.*
- For each element $e \in E$, a *continuous* and *nondecreasing* delay function $d_e : \mathbb{R}_+ \to \mathbb{R}_+$.

Given a feasible flow $\boldsymbol{f}$, we define the *load* of element e:

$$x_e := \sum_{\{P\in\mathcal{S}:\ e\in P\}} f_P$$

and the cost for an (infinitesimal) player choosing strategy $P \in \mathcal{S}$ is $c_P(\boldsymbol{x}) := \sum_{e\in P} d_e(x_e)$. We denote by X the set of all load vectors induced by a feasible flow : $X = \big\{\boldsymbol{x} :\ \exists \boldsymbol{f} \text{ feasible s.t. } x_e = \sum_{P\ni e} f_P\big\}$.

Since the players are infinitesimal, the standard Nash equilibrium is not well defined for this class of game. By taking the limit of the characterization of a NE for atomic games (cf. Proposition **??**), we obtain the following definition:

Definition 3 (Wardrop Equilibrium)**.** A feasible flow $\boldsymbol{f}$ for a non-atomic congestion game $(E, \mathcal{S}, \boldsymbol{w}, \boldsymbol{d})$ is a *Wardrop equilibrium* if and only if its induced load $\boldsymbol{x}$ satisfies:

$$\forall i \in [N],\ \forall (P,Q) \in S^i \times S^i \text{ s.t. } f_P > 0, \quad \sum_{e\in P} d_e(x_e) \leq \sum_{e\in Q} d_e(x_e).$$

In other words, the flow $\boldsymbol{f}$ only assigns weight to minimum-cost strategies P (i.e., P minimizes $c_P(\boldsymbol{x})$ over some S^i for the current load $\boldsymbol{x}$). This reflects the fact that no infinitesimal player has an incentive to change (unilaterally) her strategy.

Definition 4 (Social optimum)**.** A feasible flow $\boldsymbol{f}$ is a called a *social optimum* of the non-atomic congestion game $(E, \mathcal{S}, \boldsymbol{w}, \boldsymbol{d})$ if its induced load $\boldsymbol{x}$ minimizes the total cost $C(\boldsymbol{x}) := \sum_{e\in E} x_e d_e(x_e)$ over X.

Proposition 4. *Consider a non-atomic congestion game* $(E, \mathcal{S}, \boldsymbol{w}, \boldsymbol{d})$ *such that for all* $e \in E$, *the function* $x \to x d_e(x)$ *is convex and continutinuously differentiable (i.e., of class* $\mathcal{C}^1$*), and define the marginal cost* $\hat{d}_e(x) = \frac{d}{dx}\Big(x d_e(x)\Big) = d_e(x) + x d_e'(x)$. *Then,* $\boldsymbol{f}$ *is a social optimum for* $(E, \mathcal{S}, \boldsymbol{w}, \boldsymbol{d})$ *iff* $\boldsymbol{f}$ *is a Wardrop equilibrium of the non-atomic congestion game* $(E, \mathcal{S}, \boldsymbol{w}, \hat{\boldsymbol{d}})$.

Proof. cf. Exercise 3 of Worksheet #6 □

We now obtain a characterization of Wardrop equilibriums based on a potential function, similarly to what was done for unweighted atomic games:

Theorem 5. *A feasible flow* $\boldsymbol{f}$ *is a Wardrop equilibrium of the non-atomic congestion game* $(E, \mathcal{S}, \boldsymbol{w}, \boldsymbol{d})$ *if and only if its induced flow* $\boldsymbol{x}$ *minimizes the convex potential* $\Phi(\boldsymbol{x}) := \sum_{e\in E} \int_0^{x_e} d_e(z)dz$ *over* X.

Proof. Define $h_e(x) = \frac{1}{x}\int_0^x d_e(z)dz$ for all $x > 0$, and extend the definition to $x = 0$ by continuity, by setting $h_e(0) = d_e(0)$. It is clear that $x \to x h_e(x)$ is convex and continutinuously differentiable, because d_e is continuous and nondecreasing. Moreover it is easy to see that h_e itself is nondecreasing ($h_e(x)$ is the average of the nondecreasing function d_e over the interval $[0, x]$). So $(E, \mathcal{S}, \boldsymbol{w}, \boldsymbol{h})$ is a non-atomic congestion game. Now, note that the marginal cost associated to h_e is $\hat{h}_e(x) = (x h_e(x))' = d_e(x)$, so we obtain the desired result by applying Proposition **??**. The convexity of Φ follows from the fact that each d_e is nondecreasing. □

Theorem 6. *Let* $(E, \mathcal{S}, \boldsymbol{w}, \boldsymbol{d})$ *be a non-atomic congestion game. Then:*

(i) This game admits at least one Wardrop Equilibrium.

(ii) If $\boldsymbol{f}$ *and* $\tilde{\boldsymbol{f}}$ *are equilibriums, inducing respectively the loads* $\boldsymbol{x}$ *and* $\tilde{\boldsymbol{x}}$, *then* $d_e(x_e) = d_e(\tilde{x}_e)$ *for all* $e \in E$. *In particular, if a delay function* d_e *is (strictly) increasing, then* $x_e = \tilde{x}_e$ *(unicity of the Wardrop load on element* e*).*

Proof. (i) We will show in Exercise 5 of Worksheet #6 that the set X of feasible loads is compact and convex. Thus, the continuous function Φ has a minimizer over X. (ii) Let $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$ be the loads associated with two Wardrop equilibriums. By convexity of X, the load $\lambda\boldsymbol{x} + (1-\lambda)\tilde{\boldsymbol{x}}$ is feasible for all $\lambda \in [0, 1]$ (i.e., induced by a feasible flow), and by convexity of Φ we have

$$\Phi(\lambda\boldsymbol{x} + (1-\lambda)\tilde{\boldsymbol{x}}) \leq \lambda\Phi(\boldsymbol{x}) + (1-\lambda)\Phi(\tilde{\boldsymbol{x}}).$$

This inequality must be an equality for all $\lambda \in [0, 1]$ (otherwise it would contradict the optimality of $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$). Now, observe that Φ is the sum of some convex functions φ_e:

$$\Phi(\boldsymbol{x}) = \sum_{e\in E} \underbrace{\int_0^{x_e} d_e(z)dz}_{\varphi_e(\boldsymbol{x})}.$$

The functions φ_e sum to a constant over the segment $[\boldsymbol{x}, \tilde{\boldsymbol{x}}]$. So over this segment, each φ_e can be rewritten as a constant minus a convex function, i.e., a concave function. So, φ_e is both convex and concave over the segment $[\boldsymbol{x}, \tilde{\boldsymbol{x}}]$, hence it must be linear. In particular, the real function $x_e \to \int_0^{x_e} d_e(z)dz$ is linear over $[x_e, \tilde{x}_e]$, which implies that d_e is constant over this interval.

□