Lecture #2 Notes Summary

Cone $\mathbb{S}_+^n$, generalized inverses

Semidefinite positive matrices and generalized inverses

Definition 1 (cone $\mathbb{S}_+^n$)**.** A symmetric matrix A is called positive semidefinite if $\boldsymbol{x}^{\boldsymbol{T}} A \boldsymbol{x} \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$, and positive definite if $\boldsymbol{x}^{\boldsymbol{T}} A \boldsymbol{x} > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0} \in \mathbb{R}^n$. The set of $n \times n$ symmetric matrices (resp. positive semidefinite, positive definite) is $\mathbb{S}^n$ (resp. $\mathbb{S}_+^n, \mathbb{S}_{++}^n$).

Proposition 1. *The set $\mathbb{S}_+^n$ is a closed, convex, pointed cone.*

We recall that the eigenvalues of a symmetric matrix are real, and any symmetric matrix can be diagonalized by an orthogonal transformation (that is, a kind of rotation):

Theorem 2 (Spectral decomposition)**.** *Let A be a symmetric matrix. Then, there exists an orthogonal matrix U (i.e., $U^T U = UU^T = \boldsymbol{I}$) such that*

$$A = U \Sigma U^T,$$

where the matrix Σ is diagonal and contains the eigenvalues $\lambda_i \in \mathbb{R}$ of A.

Definition 2 (Löwner ordering)**.** The space $\mathbb{S}^n$ is equipped with a partial ordering $\succeq$, defined as

$$A \succeq B \iff A - B \in \mathbb{S}_+^n.$$

In particular, we write $A \succeq 0$ if A is positive semidefinite (and $A \succ 0$ if A is positive definite).

Characterization of positive semidefinite matrices:

Theorem 3. *The following statements are equivalent*

(i) $A \in \mathbb{S}_+^n$.

(ii) All eigenvalues of A are ≥ 0.

(iii) $A \in \text{convex-hull}\{\boldsymbol{x}\boldsymbol{x}^T,\ \boldsymbol{x} \in \mathbb{R}^n\}$.

(iv) There exists a $n \times n-$matrix B such that $A = BB^T$.

and the equivalent theorem for positive definite matrices:

Theorem 4. *The following statements are equivalent*

(i) $A \in \mathbb{S}_{++}^n$.

(ii) All eigenvalues of A are > 0.

(iii) There exists an invertible $n \times n-$matrix B such that $A = BB^T$.

Some useful properties

Proposition 5 (Congruent transformations)**.** *Let P be an $m \times n-$matrix. Then,*

$$A \succeq 0 \Longrightarrow PAP^T \succeq 0.$$

Moreover, if P is square and invertible the converse relation holds ($PAP^T \succeq 0 \Longrightarrow A \succeq 0$).

Proposition 6. *Let $A \in \mathbb{S}_+^n$. Then,*

- *All principal submatrices of A are $\succeq 0$;*
- *All diagonal elements of A are ≥ 0;*
- $\forall i, j,\ |A_{ij}| \leq \sqrt{A_{ii}A_{jj}} \leq \frac{1}{2}(A_{ii} + A_{jj})$.

Proposition 7 (Schur complement)**.** *Let $A \succ 0$. Then the following holds:*

$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \succeq 0 \Longleftrightarrow C \succeq B^T A^{-1} B.$$

There is also use a stronger version of this proposition, for the case where we only have $A \succeq 0$. But this requires the concept of *generalized inverses*.

Definition 3 (Generalized inverse)**.** Let M be an $n \times m-$matrix. The matrix G is called a *generalized inverse* of M if $MGM = M$.

Proposition 8. *Let $M \in \mathbb{S}^n$. Let K_1 and K_2 be matrices whose colums are included in the range of M (i.e., $\operatorname{im} K_1 \subseteq \operatorname{im} M$, $\operatorname{im} K_2 \subseteq \operatorname{im} M$). Then, the matrix $K_1^T G K_2$ does not depend on the choice of a generalized inverse G of M. We denote this matrix by $K_1^T M^- K_2$.*

Definition 4 (Moore-Penrose pseudoinverse)**.** Let M be an $n \times m-$matrix. The matrix

$$M^\dagger := \lim_{\epsilon \to 0^+} (M^T M + \epsilon \boldsymbol{I}_m)^{-1} M^T$$

is a generalized inverse of M, called the *Moore-Penrose pseudoinverse* of M. In practice, $M^\dagger$ can be computed by taking a singular value decomposition of M, and inverting the nonzero singular values.

Proposition 9 (Generalized Schur complement)**.** *Let $A \succeq 0$, and let B be a matrix satisfying the range inclusion condition* $\operatorname{im} B \subseteq \operatorname{im} A$. *Then the following holds:*

$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \succeq 0 \Longleftrightarrow C \succeq B^T A^- B.$$

Exercises

1. Let $A \succ 0$. Show that the set $E = \{\boldsymbol{x} : \boldsymbol{x}^T A \boldsymbol{x} \leq 1\}$ is an ellipsoid. More precisely, show that E is the image by an orthogonal transformation of a set of the form

$$\{\boldsymbol{y} : \sum_i \left(\frac{yi}{r_i}\right)^2 \leq 1\}.$$

What are the semi-axis of E ?

2. Let $A \succeq 0$ and $B \succeq 0$, and consider the ellipsoids $E_A = \{\boldsymbol{x} : \boldsymbol{x}^T A \boldsymbol{x} \leq 1\}, E_B = \{\boldsymbol{x} : \boldsymbol{x}^T B \boldsymbol{x} \leq 1\}$. Show the equivalence

$$E_A \subseteq E_B \Longleftrightarrow A \succeq B.$$

3. Let A be a $n \times n-$symmetrix matrix, satifying $A \preceq \lambda \boldsymbol{I}$. Show that we have $\lambda \geq \lambda_{max}$, where λ_{max} is the largest eigenvale of A. Show that $A \preceq \lambda_{max} \boldsymbol{I}_n$.

4. Let $A \succ 0$, and B be an arbitrary matrix. Use a congruence tranformation with the matrix $\begin{pmatrix} \boldsymbol{I} & 0 \\ B^T A^{-1} & \boldsymbol{I} \end{pmatrix}$ to proove the Schur complement lemma.

5. Let $A \succeq 0$. Show that A admits a square root, i.e., a matrix $X \succeq 0$ such that $A = X^2$.

6. (Geometric mean of two positive definite matrices). Let $A, B \succ 0$. We admit that for all $Y, Z \succeq 0$, $Y \preceq Z \Longrightarrow Y^{1/2} \preceq Z^{1/2}$ (NB: the converse is not true !). Let X be a symmetrix matrix satisfying $\begin{pmatrix} A & X \\ X & B \end{pmatrix} \succeq 0$. Show that

$$X \preceq A \# B := A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}.$$