Supplemental Material on "Depinning transition of self-propelled particles"

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I. NUMERICS OF THE ACTIVE BROWNIAN PARTICLE (ABP) MODEL

A. Stochastic simulation of the Itō–Langevin equations

For the stochastic simulation of the ABP model given by Eqs. (1b) and (2) in the main text, we have generated, for each force f, up to 5×10^5 random trajectories $x(t)$ of length $5 \times 10^4 \tau_\mathrm{L}$. To this end, we combined the Euler(–Maruyama) integration for the translational motion and a geometric integration scheme[1] for the rotational Brownian motion, using an integration time step of $\Delta t = 10^{-3}\tau_\mathrm{L}$. In addition, we have applied a simple antithetic variance reduction technique, where for every noise realization $\boldsymbol{\omega}(t)$ one obtains two trajectories: one with $\boldsymbol{\omega}(t)$ and one with $-\boldsymbol{\omega}(t)$, exploiting the inflection symmetry of the noise.

B. Numerical solution of the Fokker–Planck equation

The Fokker–Planck equation (FPE) corresponding to the Itō–Langevin Eqs. (1a) and (1b) of the main text reads

$$\partial_t p(\boldsymbol{r}, \boldsymbol{u}, t) = -\nabla \cdot [\mu_0(\boldsymbol{f} - \nabla U(\boldsymbol{r})) + v_\mathrm{A}\boldsymbol{u}]\, p(\boldsymbol{r}, \boldsymbol{u}, t) + \tau_\mathrm{R}^{-1} L_{\boldsymbol{u}}\, p(\boldsymbol{r}, \boldsymbol{u}, t)\,, \tag{S1}$$

where $p(x, \boldsymbol{u}, t)$ is the joint probability density of the position x and the orientation $\boldsymbol{u}$ at time t and $L_{\boldsymbol{u}}$ denotes the Laplace–Beltrami operator on the d-dimensional unit sphere. For the one-dimensional corrugated potential landscape discussed in this work, only the projection $x = \boldsymbol{r} \cdot \boldsymbol{e}_x$ and the polar angle ϑ such that $z := \cos(\vartheta) = \boldsymbol{u} \cdot \boldsymbol{e}_x$ are relevant. Then, Eq. (S1) reduces to

$$\partial_t p(x, z, t) = -\partial_x[\mu_0 f - v_\mathrm{L}\sin(kx) + v_\mathrm{A} z]\, p(x, z, t) + \tau_\mathrm{R}^{-1}\partial_z\big(1 - z^2\big)\partial_z\, p(x, z, t)\,, \tag{S2}$$

which is the FPE corresponding to Eqs. (1b) and (2) of the main text. The domain of $p(x, z, t)$ is $x \in \mathbb{R}$, $z \in [-1, 1]$, and $t \geqslant 0$.

Exploiting the inherent x-periodicity of the problem, we proceed to the reduced probability density[2] $\hat{p}(x, z, t) := \sum_{n=-\infty}^{\infty} p(x + n\lambda, z, t)$, which satisfies Eq. (S2) for $x \in [0, \lambda]$ with periodic boundary conditions, $\hat{p}(x, z, t) = \hat{p}(x + \lambda, z, t)$ with $\lambda = 2\pi/k$. We recall further that the eigenfunctions of the $d = 3$ rotational diffusion operator are the Legendre polynomials $P_\ell(z)$, i.e.,

$$\partial_z\big(1 - z^2\big)\partial_z P_\ell(z) = -\ell(\ell + 1)P_\ell(z)\,; \qquad \ell \in \mathbb{N}_0\,. \tag{S3}$$

The periodicity of $\hat{p}(x, z, t)$ in x together with Eq. (S3) suggest to represent the solution as a Fourier–Legendre series,

$$\hat{p}(x, z, t) = \sum_{n \in \mathbb{Z}} \sum_{\ell \geqslant 0} c_{n\ell}(t) \mathrm{e}^{\mathrm{i}nkx} P_\ell(z)\,. \tag{S4}$$

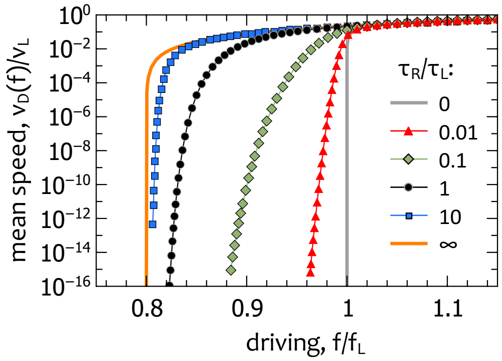

FIG. S1. Drift velocity $v_{\mathrm{D}}(f)$ as function of the driving force f was obtained from the numerical FPE solution [Eq. (S6)] with the self-propulsion velocity fixed to $v_{\mathrm{A}} = 0.2v_{\mathrm{L}}$. The same data are shown in Fig. 1c of the main text on a super-logarithmic scale.

The time evolution of the coefficients $c_{n\ell}(t)$ is implied by Eq. (S2) and one finds:

$$\dot{c}_{n0} = -\,\mathrm{i}nk\left[\mu_0 f c_{n0} - \frac{v_{\mathrm{L}}}{2\mathrm{i}}(c_{n-1,0} - c_{n+1,0})\right] - \mathrm{i}nkv_{\mathrm{A}}\frac{c_{n1}}{3}\,; \qquad \ell = 0, \tag{S5a}$$

$$\begin{aligned}\dot{c}_{n\ell} = &-\,\mathrm{i}nk\left[\mu_0 f c_{n\ell} - \frac{v_{\mathrm{L}}}{2\mathrm{i}}(c_{n-1,\ell} - c_{n+1,\ell})\right]\\ &-\,\mathrm{i}nkv_{\mathrm{A}}\left(\frac{\ell}{2\ell-1}c_{n,\ell-1} + \frac{\ell+1}{2\ell+3}c_{n,\ell+1}\right) - \tau_{\mathrm{R}}^{-1}\ell(\ell+1)c_{n\ell}\,; \qquad \ell > 0.\end{aligned} \tag{S5b}$$

For the stationary solution, the left hand sides are set to zero, $\dot{c}_{n\ell} = 0$, and Eq. (S5) becomes a linear system in the coefficients $c_{n\ell}$. The normalization condition $\int \hat{p}(x,z,t)\,\mathrm{d}x\mathrm{d}z = 1$ implies $c_{00} = 1/2$, which renders the linear system inhomogeneous. We truncated the series (S4) symmetrically to keep only terms with $-N \leqslant n \leqslant N$ and $0 \leqslant \ell \leqslant L$ and solved the system of $(2N+1)\times(L+1)$ equations numerically using standard BLAS routines.

The mean speed $v_{\mathrm{D}}(f) = \lim_{t\to\infty}\langle\dot{x}(t)\rangle$ is the integral of the x-component of the probability flux $\langle\dot{x}(t)\rangle = \int \hat{\jmath}_x(x,z,t)\,\mathrm{d}x\mathrm{d}z$ with $\hat{\jmath}_x(x,z,t) = (\mu_0 f - v_{\mathrm{L}}\sin(kx) + v_{\mathrm{A}}z)\hat{p}(x,z,t)$ and, upon using Eq. (S4), it is calculated from the expansion coefficients as

$$v_{\mathrm{D}}(f) = \mu_0 f + 2v_{\mathrm{L}}\,\mathrm{Im}\,c_{1,0} + \frac{2}{3}v_{\mathrm{A}}c_{0,1}\,. \tag{S6}$$

The numerical results shown in Fig. S1 and in Fig. 1(c) of the main text were obtained for $N = 10\,000$ and $L = 30$. The different orders of magnitude for N and L were chosen to account for the observation that the eigenvalues of ∇ scale as n, whereas the eigenvalues of $L_{\boldsymbol{u}}$ scale as $\ell(\ell+1)$.

II. DRIFT VELOCITY AND DISPERSION COEFFICIENT OF LAZY WOBBLERS

A. Random walk model

As described in the main text, the trajectories $x(t)$ in the regime of the lazy wobbler ($\tau_\mathrm{R} \gg \tau_\mathrm{L}$ and $\tau_R \gg \tau_f$) are approximated by a one-dimensional random walk (or "flight") such that the orientation of the particle changes instantaneously at random times with a rate τ_R^{-1}. In this heuristic model, the orientation $\boldsymbol{u}(t)$ consists of piecewise constant segments $\boldsymbol{u}_i$ of random durations τ_i for $i = 1, 2, \dots$. For the depinning problem, we may equivalently use the angles ϑ_i such that $\cos\vartheta_i = \boldsymbol{u}_i \cdot \boldsymbol{e}_x$. Given a fixed orientation $\boldsymbol{u}_i$ (or ϑ_i), the particle moves at the velocity $v_\mathrm{D}(f;\vartheta_i)$ for a time span τ_i. Then, assuming $x(0) = 0$, the spatial displacement after time t is

$$x(t) = \sum_{i=1}^{N(t)} v_\mathrm{D}(f;\vartheta_i)\tau_i\,, \tag{S7}$$

where $N(t)$ is counting the reorientation events up to and including time $t = \sum_{i=1}^{N(t)} \tau_i$. The resulting trajectories $x(t)$ correspond exactly to the motion of run-and-tumble particles.

Following the ideas of Boltzmann's Stoßzahlansatz (molecular chaos hypothesis)[3], the reorientation events ("collisions") are assumed to be independent and combine exponentially distributed times τ_i between subsequent collisions with orientations $\boldsymbol{u}_i$ that are sampled independently from the equilibrium distribution, i.e., a uniform distribution on the unit sphere, $|\boldsymbol{u}| = 1$. As a consequence, $N(t)$ is a Poisson process with parameter τ_R^{-1}.

B. Drift velocity

For the drift velocity (or: mean speed), one finds from Eq. (S7):

$$\begin{aligned} v_\mathrm{D}^{(\infty)}(f) &= \lim_{t\to\infty} \frac{x(t)}{t} \\ &= \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} v_\mathrm{D}(f;\vartheta_i)\tau_i \Big/ \frac{1}{N}\sum_{i=1}^{N}\tau_i \\ &= \frac{\langle v_\mathrm{D}(f;\vartheta_i)\rangle_{\boldsymbol{u}} \langle \tau_i\rangle}{\langle\tau_i\rangle} \\ &= \frac{1}{4\pi}\int v_\mathrm{D}(f;\vartheta)\sin(\vartheta)\,\mathrm{d}\vartheta\mathrm{d}\varphi\,. \end{aligned} \tag{S8}$$

In the second line, we have used that $N(t\to\infty)\to\infty$ monotonically, which permits that the limit $t\to\infty$ is replaced by letting $N\to\infty$. The third line follows from the strong law of large numbers and the independence of ϑ_i and τ_i. The last line of Eq. (S8) represents the orientation-averaged drift velocity, $\langle v_\mathrm{D}(f;\vartheta_i)\rangle_{\boldsymbol{u}}$. We rewrite the integrand, as in the main text, in terms of $v_\mathrm{D}(f;\vartheta) = v_\mathrm{L} s(f_\mathrm{A}(\vartheta)/f_\mathrm{L})$ with the effective driving force $f_\mathrm{A}(\vartheta) = f + (v_\mathrm{A}/\mu_0)\cos\vartheta$ and $s(y) = \sqrt{y^2-1}$ for $|y| > 1$ and $s(y) = 0$ otherwise. Substituting $u_x = \cos\vartheta$, the $\boldsymbol{u}$-average is calculated as:

$$v_\mathrm{D}^{(\infty)}(f) = \frac{v_\mathrm{L}}{2}\int_{-1}^{1} s\big(f/f_\mathrm{L} + (v_\mathrm{A}/v_\mathrm{L})u_x\big)\,\mathrm{d}u_x = \frac{v_\mathrm{L}^2}{2v_\mathrm{A}}\int_{y_-}^{y_+} s(y)\,\mathrm{d}y\,, \tag{S9}$$

after substituting $y = f_\mathrm{A}(\vartheta)/f_\mathrm{L} = f/f_\mathrm{L} + (v_\mathrm{A}/v_\mathrm{L})u_x$ for u_x with the new integral bounds $y_\pm = f/f_\mathrm{L} \pm v_\mathrm{A}/v_\mathrm{L}$. The remaining integral is elementary:

$$w(y) := \int_0^y s(y')\,\mathrm{d}y' = \frac{1}{2}[ys(y) - \ln\big(y + s(y)\big)] \quad \text{for } y > 1, \tag{S10}$$

and $w(y) = 0$ otherwise. Introducing $w_\pm(f/f_\mathrm{L}) := w(y_\pm) = w(y \pm v_\mathrm{A}/v_\mathrm{L})$, we obtain the result quoted in Eq. (8) of the main text:

$$v_\mathrm{D}^{(\infty)}(f) = \frac{v_\mathrm{L}^2}{2v_\mathrm{A}} \big[w_+(f/f_\mathrm{L}) - w_-(f/f_\mathrm{L})\big]. \tag{S11}$$

Inserting the piecewise expressions of $w(y)$ for $y \leqslant 1$ and $y > 1$, respectively, yields three explicit cases:

$$v_\mathrm{D}^{(\infty)}(f) = \frac{v_\mathrm{L}^2}{2v_\mathrm{A}} \begin{cases} 0, & f \leqslant f_\mathrm{c}^-, \\ w_+(f/f_\mathrm{L}), & f_\mathrm{c}^- < f < f_\mathrm{c}^+, \\ w_+(f/f_\mathrm{L}) - w_-(f/f_\mathrm{L}), & f \geqslant f_\mathrm{c}^+. \end{cases} \tag{S12}$$

C. Critical behavior

To obtain the critical behavior of the drift velocity, we introduce the distance to the critical point, $\varepsilon = (f - f_\mathrm{c}^-)/f_\mathrm{L}$, and find the leading term in an asymptotic expansion of the integral in Eq. (S9). Restricting to $0 < \varepsilon < 2v_\mathrm{A}/v_\mathrm{L}$, it holds $y_+ = 1+\varepsilon$ and $y_- = 1+\varepsilon-2v_\mathrm{A}/v_\mathrm{L} < 1$, which simplifies the integral bounds. Introducing a new integration variable $0 \leqslant \eta \leqslant 1$ such that $y = 1 + \eta\varepsilon$ yields:

$$\begin{aligned} v_\mathrm{D}^{(\infty)}(f = f_\mathrm{c}^- + \varepsilon f_\mathrm{L}) &= \frac{v_\mathrm{L}^2}{2v_\mathrm{A}} \int_1^{1+\varepsilon} s(y)\,\mathrm{d}y = \frac{v_\mathrm{L}^2}{2v_\mathrm{A}}\, \varepsilon \int_0^1 \sqrt{2\eta\varepsilon}\,[1 + O(\varepsilon)]\,\mathrm{d}\eta \\ &= \frac{v_\mathrm{L}^2}{2v_\mathrm{A}} \frac{2\sqrt{2}}{3}\, \varepsilon^{3/2}\,[1 + O(\varepsilon)]\,. \end{aligned} \tag{S13}$$

We used that $s(y)$ is bounded on the domain of integration, which permits interchanging the η-integral with the expansion for $\varepsilon \to 0$. Hence,

$$v_\mathrm{D}^{(\infty)}(f \downarrow f_\mathrm{c}^-) \simeq \frac{\sqrt{2}}{3} \frac{v_\mathrm{L}^2}{v_\mathrm{A}}\, \varepsilon^{3/2} \sim (f - f_\mathrm{c}^-)^{3/2}\,. \tag{S14}$$

Alternatively, the same result is obtained by expanding $w_+(1 + \varepsilon)$ defined after Eq. (S10).

D. Extension to rotational motion in the plane

The preceding analysis of the lazy-wobbling limit has a straightforward extension to ABP models with two-dimensional rotational motion, where the self-propulsion velocity is constrained to the plane of translational motion. The essential difference is that the orientation vector $\boldsymbol{u}$ is uniformly distributed on a circle rather than on a sphere, which has implications for the integrals implementing the $\boldsymbol{u}$-average. For the mean speed, Eq. (S8) is replaced by

$$v_\mathrm{D}^{(\infty)}(f) = \frac{1}{\pi} \int_0^\pi v_\mathrm{D}(f; \vartheta)\,\mathrm{d}\vartheta\,, \tag{S15}$$

where we stick to a representation in terms of the polar angle $\vartheta \in [0, \pi]$. Relative to Eq. (S8), the factor $\sin(\vartheta)$ is missing from the differential of the solid angle. Nevertheless, we substitute $u_x = \cos(\vartheta)$ with $\mathrm{d}u_x = \sin(\vartheta)\,\mathrm{d}\vartheta = \sqrt{1 - u_x^2}\,\mathrm{d}\vartheta$ and, subsequently, introduce y as above. With this, the expression corresponding to Eq. (S9) reads

$$\begin{aligned} v_\mathrm{D}^{(\infty)}(f) &= \frac{v_\mathrm{L}}{\pi} \int_{-1}^1 \frac{s\big(f/f_\mathrm{L} + (v_\mathrm{A}/v_\mathrm{L})u_x\big)}{\sqrt{1 - u_x^2}}\,\mathrm{d}u_x \\ &= \frac{v_\mathrm{L}^2}{\pi v_\mathrm{A}} \int_{y_-}^{y_+} \frac{s(y)}{\sqrt{1 - r(y)^2}}\,\mathrm{d}y\,, \end{aligned} \tag{S16}$$

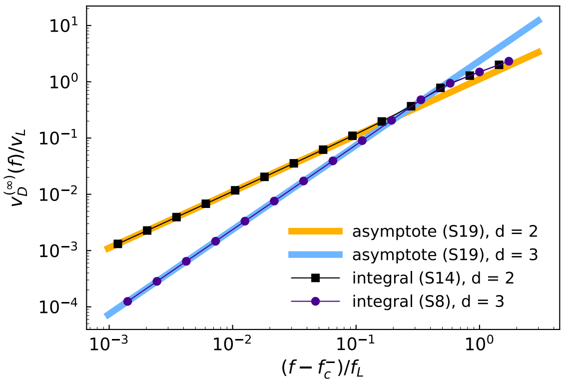

FIG. S2. Critical behavior of the drift velocity $v_{\mathrm{D}}^{(\infty)}(f)$ in the lazy-wobbling limit ($\tau_{\mathrm{R}} \to \infty$) as function of the distance to the critical point, $\varepsilon = (f - f_{\mathrm{c}}^-)/f_{\mathrm{L}}$, evaluated for $v_{\mathrm{A}}/v_{\mathrm{L}} = 0.2$ and for rotational motion in $d = 2$ and $d = 3$ dimensions. Solid lines show the analytic prediction for both cases [Eq. (S20)] and symbols denote numerical results from the quadrature of the orientational $\boldsymbol{u}$-average given in Eq. (S15) for $d = 2$ (squares) and Eq. (S8) for $d = 3$ (circles); the latter agree also with the explicit expression in Eq. (S11).

upon replacing $u_x = r(y) := (v_{\mathrm{L}}/v_{\mathrm{A}})(y - f/f_{\mathrm{L}})$ by y and $y_\pm = f/f_{\mathrm{L}} \pm v_{\mathrm{A}}/v_{\mathrm{L}}$, as before.

In the absence of an explicit form for the integral in Eq. (S16), we determine the critical behavior close to the critical point analogously as above for $d = 3$. Writing again $f = f_{\mathrm{c}}^- + \varepsilon f_{\mathrm{L}}$, it holds $r(y;\varepsilon) = 1 + (v_{\mathrm{L}}/v_{\mathrm{A}})(y - 1 - \varepsilon)$. For $0 < \varepsilon < 2v_{\mathrm{A}}/v_{\mathrm{L}}$, we thus have

$$v_{\mathrm{D}}^{(\infty)}(f = f_{\mathrm{c}}^- + \varepsilon f_{\mathrm{L}}) = \frac{v_{\mathrm{L}}^2}{\pi v_{\mathrm{A}}} \int_1^{1+\varepsilon} \frac{s(y)}{\sqrt{1 - r(y;\varepsilon)^2}}\,\mathrm{d}y\,. \tag{S17}$$

Passing on to the integration variable η such that $y = 1 + \eta\varepsilon$, the leading order in ε is obtained by letting $\varepsilon \to 0$ in the integrand:

$$\begin{aligned} v_{\mathrm{D}}^{(\infty)}(f = f_{\mathrm{c}}^- + \varepsilon f_{\mathrm{L}}) &= \frac{v_{\mathrm{L}}^2}{\pi v_{\mathrm{A}}} \int_0^1 \frac{\sqrt{2\eta\varepsilon + O(\varepsilon^2)}}{\sqrt{2(v_{\mathrm{L}}/v_{\mathrm{A}})(1-\eta)\varepsilon + O(\varepsilon^2)}}\,\varepsilon\mathrm{d}\eta \\ &\simeq \frac{v_{\mathrm{L}}^{3/2}\varepsilon}{\pi v_{\mathrm{A}}^{1/2}} \int_0^1 \sqrt{\eta/(1-\eta)}\,\mathrm{d}\eta \\ &= \frac{v_{\mathrm{L}}^{3/2}\varepsilon}{2 v_{\mathrm{A}}^{1/2}}\,; \end{aligned} \tag{S18}$$

the integral in the last step evaluates to $\pi/2$. Thus close to the critical point, it holds for the $d = 2$ case:

$$v_{\mathrm{D}}^{(\infty)}(f \downarrow f_{\mathrm{c}}^-) \simeq \frac{1}{2}(v_{\mathrm{L}}/v_{\mathrm{A}})^{1/2}\mu_0(f - f_{\mathrm{c}}^-) \sim f - f_{\mathrm{c}}^-\,. \tag{S19}$$

The critical laws in Eqs. (S14) and (S19) can be summarized for $d = 2, 3$ as

$$v_{\mathrm{D}}^{(\infty)}(f \downarrow f_{\mathrm{c}}^-) \simeq \frac{1}{d}\big[(d-1)v_{\mathrm{L}}v_{\mathrm{A}}\big]^{1/2}\,\big[\mu_0(f - f_{\mathrm{c}}^-)/v_{\mathrm{A}}\big]^{d/2}\,. \tag{S20}$$

Figure S2 corroborates this analytic result, which coincides asymptotically ($f \downarrow f_{\mathrm{c}}^-$) with the data for $v_{\mathrm{D}}^{(\infty)}(f)$ from the quadrature of the $\boldsymbol{u}$-average [Eq. (S8) for $d = 3$, Eq. (S15) for $d = 2$].

E. Dispersion coefficient

Concerning the dispersion of the trajectories, we note first that the sequence of reorientations yields, in full analogy to the particle collisions in a dilute gas, for the velocity autocorrelation function[3]:

$$\begin{aligned} Z(t) &= \langle [v_\mathrm{D}(f;\vartheta(t)) - v_\mathrm{D}^{(\infty)}(f)]\,[v_\mathrm{D}(f;\vartheta(0)) - v_\mathrm{D}^{(\infty)}(f)]\rangle \\ &= \mathrm{Var}[v_\mathrm{D}(f;\vartheta_i)]_{\boldsymbol{u}}\,\mathrm{e}^{-t/\tau_\mathrm{R}}\,, \end{aligned} \tag{S21}$$

taking into account that $\langle v_\mathrm{D}(f;\vartheta_i)\rangle_{\boldsymbol{u}} = v_\mathrm{D}^{(\infty)}(f)$ may not be zero. The factor $\mathrm{e}^{-t/\tau_\mathrm{R}}$ is simply the probability that no "collision" has occurred in the time interval $[0,t]$. The effective diffusion coefficient then follows from the Green–Kubo relation:

$$\begin{aligned} D_\mathrm{eff}(f) &= \int_0^\infty Z(t)\,\mathrm{d}t = \mathrm{Var}[v_\mathrm{D}(f;\vartheta_i)]_{\boldsymbol{u}}\tau_\mathrm{R} \\ &= \big(\langle v_\mathrm{D}(f;\vartheta_i)^2\rangle_{\boldsymbol{u}} - v_\mathrm{D}^{(\infty)}(f)^2\big)\tau_\mathrm{R}\,. \end{aligned} \tag{S22}$$

It remains to compute the second moment, $\langle v_\mathrm{D}(f;\vartheta_i)^2\rangle_{\boldsymbol{u}}$. The same arguments apply that have led to Eq. (S9) for the first moment in the case $d=3$. Therefore:

$$\begin{aligned} \langle v_\mathrm{D}(f;\vartheta_i)^2\rangle_{\boldsymbol{u}} &= \frac{v_\mathrm{L}^2}{2}\int_{-1}^{1} s(f/f_\mathrm{L} + (v_\mathrm{A}/v_\mathrm{L})u_x)^2\,\mathrm{d}u_x \\ &= \frac{v_\mathrm{L}^3}{2v_\mathrm{A}}\int_{\max(y_-,1)}^{\max(y_+,1)} \big(y^2-1\big)\,\mathrm{d}y\,. \end{aligned} \tag{S23}$$

where the integral bounds in the last line have been tightened to the condition $|y|>1$. Introducing $\tilde{w}_\pm(f/f_\mathrm{L}) = \tilde{w}(y_\pm)$ with

$$\tilde{w}(y) := \int_0^y s(y')^2\,\mathrm{d}y' = \frac{1}{3}\big(y^3-1\big) + 1 - y \quad \text{if } y>1 \tag{S24}$$

and $\tilde{w}(y)=0$ otherwise, it follows

$$\langle v_\mathrm{D}(f;\vartheta_i)^2\rangle_{\boldsymbol{u}} = \frac{v_\mathrm{L}^3}{2v_\mathrm{A}}\big[\tilde{w}_+(f/f_\mathrm{L}) - \tilde{w}_-(f/f_\mathrm{L})\big]\,. \tag{S25}$$

This can be rewritten explicitly as

$$\langle v_\mathrm{D}(f;\vartheta_i)^2\rangle_{\boldsymbol{u}} = v_\mathrm{L}^2 \times \begin{cases} 0, & f \leqslant f_\mathrm{c}^-, \\ \dfrac{v_\mathrm{L}/v_\mathrm{A}}{6f_\mathrm{L}^3}\,(f-f_\mathrm{c}^-)^2(f+f_\mathrm{c}^+ + f_\mathrm{L}), & f_\mathrm{c}^- < f < f_\mathrm{c}^+, \\ (f/f_\mathrm{L})^2 + \dfrac{1}{3}(v_\mathrm{A}/v_\mathrm{L})^2 - 1, & f \geqslant f_\mathrm{c}^+. \end{cases} \tag{S26}$$

The dispersion coefficient $D_\mathrm{eff}(f)$ is obtained by inserting Eqs. (S11) and (S25) into Eq. (S22), and its behavior is exemplarily shown in Fig. 1(f) of the main text (orange line).

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