9.10.19 Mathe 2 - Übung 1

Aufgabe 1 $z = a + ib \quad z' = a' + ib'$

Summe: $z + z' = (a + ib) + (a' + ib') = (a + a') + i(b + b')$

Differenz: $z - z' = (a + ib) - (a' + ib') = a + ib - a' - ib' = (a - a') + i(b - b')$

Produkt: $z \cdot z' = (a + ib) \cdot (a' + ib') = a \cdot a' + a \cdot ib' + ib \cdot a' + i \cdot i \cdot b \cdot b'$

$= (a \cdot a' - b \cdot b') + i(ab' + ba')$

Quotient: $\frac{z}{z'} = \frac{a + ib}{a' + ib'} = \frac{(a + ib) \cdot (a' - ib')}{(a' + ib') \cdot (a' - ib')} = \frac{aa' - iab' + iba' - i^2 bb'}{a'a' - ia'b' + ib'a' - i^2 b'b'}$

$= \frac{aa' + bb' + i(ba' - ab')}{a'^2 + b'^2} = \frac{aa' + bb'}{a'^2 + b'^2} + i \cdot \frac{ba' - ab'}{a'^2 + b'^2}$

a) $z = 2 + 4i,\ w = 3 - i$

$z + w = (2 + 4i) + (3 - i) = 2 + 3 + 4i - i = 5 + 3i$

$z - w = (2 + 4i) - (3 - i) = 2 + 4i - 3 + i = -1 + 5i$

$z \cdot w = (2 + 4i) \cdot (3 - i) = 2 \cdot 3 + 2 \cdot (-i) + 4i \cdot 3 + 4i \cdot (-i)$

$= 6 - 2i + 12i + 4 = 10 + 10i$

$\frac{z}{w} = \frac{2 + 4i}{3 - i} = \frac{2 + 4i}{3 - i} \cdot \frac{3 + i}{3 + i} = \frac{6 + 2i + 12i - 4}{9 + 3i - 3i + 1} = \frac{2 + 14i}{10} = \frac{2}{10} + \frac{14}{10} i$

$= \frac{1}{5} + \frac{7}{5} i$

b) $z = -i + 5,\ w = -i$

$z + w = (-i + 5) + (-i) = 5 - 2i$

$z - w = (-i + 5) - (-i) = 5$

$z \cdot w = (-i + 5) \cdot (-i) = (-i)^2 - 5i = -1 - 5i$

$\frac{z}{w} = \frac{-i + 5}{-i} = \frac{(-i + 5)}{-i} \cdot \frac{i}{i} = \frac{1 + 5i}{1} = 1 + 5i$

c) $z = 2e^{i\frac{\pi}{3}},\ w = \frac{1}{2} e^{i\frac{\pi}{2}}$

$z = r \cdot e^{i\alpha} = \underbrace{r \cdot \cos(\alpha)}_{x} + i \cdot \underbrace{r \cdot \sin(\alpha)}_{y} = x + iy$

$z = 2 \cdot \cos\left(\frac{\pi}{3}\right) + i \cdot 2\sin\left(\frac{\pi}{3}\right) = 2 \cdot \frac{1}{2} + i \cdot 2 \cdot \frac{\sqrt{3}}{2} = 1 + i\sqrt{3}$

$w = \frac{1}{2} \cdot e^{i\frac{\pi}{2}} = \frac{1}{2} \cos\left(\frac{\pi}{2}\right) + i \cdot \frac{1}{2} \sin\left(\frac{\pi}{2}\right) = \frac{1}{2} i$

$z + w = (1 + i\sqrt{3}) + \frac{1}{2} i = 1 + \frac{1 + 2\sqrt{3}}{2} i$

$z - w = (1 + i\sqrt{3}) - \frac{1}{2} i = 1 + \frac{-1 + 2\sqrt{3}}{2} i$

$z \cdot w = (1 + i\sqrt{3}) \cdot \frac{1}{2} i = \frac{1}{2} i - \frac{\sqrt{3}}{2}$

$\frac{z}{w} = \frac{1+\sqrt{3}i}{\frac{1}{2}i} = \frac{2\cdot(1+\sqrt{3}i)}{i} = \frac{2+2\sqrt{3}i}{i} = \frac{2+2\sqrt{3}i}{i} \cdot \frac{-i}{-i}$

$= \frac{-2i+2\sqrt{3}}{1} = 2\sqrt{3} - 2i$

alternativ: $\frac{z}{w} = \frac{1+\sqrt{3}i}{\frac{1}{2}i} = \frac{1+\sqrt{3}i}{\frac{1}{2}i} \cdot \frac{-\frac{1}{2}i}{-\frac{1}{2}i} = \frac{-\frac{1}{2}i - \frac{\sqrt{3}}{2}i^2}{\frac{1}{4}}$

$= 4\cdot\left(-\frac{1}{2}i - \frac{\sqrt{3}}{2}i^2\right) = 2\sqrt{3} - 2i$

d) z = eine komplexe Zahl mit Argument 270° und Betrag 2

270° in Bogenmaß $\rightarrow 3\frac{\pi}{2}$, r = 2

$z = r\cdot e^{i\alpha} = 2\cdot e^{i\frac{3\pi}{2}} = 2\cdot\cos\left(\frac{3\pi}{2}\right) + i\cdot 2\cdot\sin\left(\frac{3\pi}{2}\right) = -2i$

w = eine komplexe Zahl mit Realteil 4 und Imaginärteil 2 $\Leftrightarrow w = 4+2i$

$z+w = -2i + (4+2i) = 4$

$z-w = -2i-(4+2i) = -2i-4-2i = -4-4i$

$z\cdot w = -2i\cdot(4+2i) = -8i-4i^2 = 4-8i$

$\frac{z}{w} = \frac{-2i}{4+2i} = \frac{-2i}{4+2i}\cdot\frac{4-2i}{4-2i} = \frac{-8i-4}{16-8i+8i+4} = \frac{-8i-4}{20} = \frac{-1}{5} - \frac{2}{5}i$

Aufgabe 2

$w^3 = \frac{8\cdot\sqrt{3}}{2} + 4i$

Hinweis: rechte Seite der Gleichung in Polarkoordinaten umschreiben

$x+iy = \underbrace{\sqrt{x^2+y^2}}_{r} \cdot e^{i\cdot\underbrace{\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\cdot \operatorname{sign}(y)}_{\alpha}} = r\cdot e^{i\alpha}$

$r = \sqrt{x^2+y^2} = \sqrt{\left(\frac{8\cdot\sqrt{3}}{2}\right)^2 + 4^2} = 8$

$\alpha = \arccos\left(\frac{\frac{8\cdot\sqrt{3}}{2}}{8}\right)\cdot \operatorname{sign}(y) = \arccos\left(\frac{\sqrt{3}}{2}\right)\cdot 1 = \frac{1}{6}\pi$

$\Rightarrow w^3 = \frac{8\cdot\sqrt{3}}{2} + 4i = 8\cdot e^{i\frac{1}{6}\pi}$

Wurzeln berechnen: $\sqrt[n]{z} = \sqrt[n]{r\cdot e^{i\alpha}} = \sqrt[n]{r}\cdot e^{i\frac{2\pi k+\alpha}{n}}$ $k=1,\ldots,n$

$w = \sqrt[3]{8\cdot e^{i\frac{\pi}{6}}} = \sqrt[3]{8}\cdot\sqrt[3]{e^{i\frac{\pi}{6}}} = \left\{2\cdot e^{i\frac{\pi}{18}}, 2\cdot e^{i\left(\frac{\pi}{18}+\frac{2\pi}{3}\right)}, 2\cdot e^{i\left(\frac{\pi}{18}+\frac{4\pi}{3}\right)}\right\}$