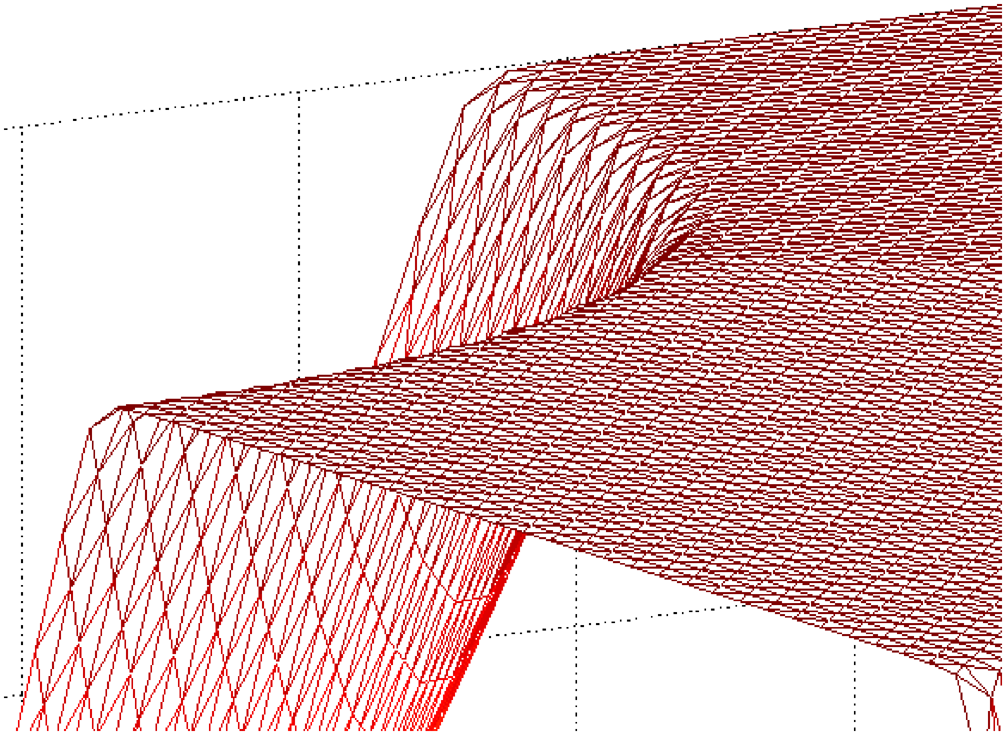


Martin Weiser

Anton Schiela
Tobias Gänzler

Primal and Primal-Dual Interior Point Methods for Optimal Control with PDEs

Optimal Control Problem

$$\min_{u\in L^\infty,\, y\in H_0^1\cap L^\infty} \frac{1}{2}\|y-y_d\|_{L_2}^2+\frac{\alpha}{2}\|u\|_{L_2}^2$$

subject to $$L\,y+u=0$$
$$-1\le u\le 1$$

$$Ly=\mathrm{div}(A\nabla y)+ay$$

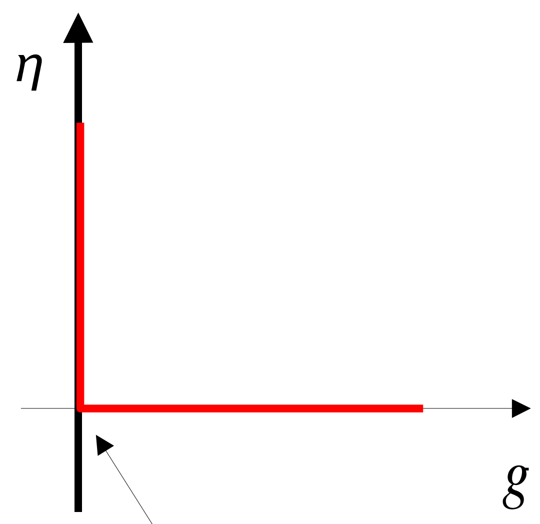
$A:\Omega\to\mathbb{R}^{2\times 2}$ uniformly spd

$a:\Omega\to\mathbb{R}$ nonnegative

L_∞ − regular

KKT conditions

$$\exists \lambda\in H_0^1\cap L^\infty,\ \underline{\eta},\overline{\eta}\in L^\infty:$$
$$y-y_d+L\lambda=0$$
$$\alpha u+\lambda-\underline{\eta}+\overline{\eta}=0$$
$$Ly+u=0$$
$$\underline{\eta}(1+u)=0$$
$$\overline{\eta}(1-u)=0$$
$$\underline{\eta},\overline{\eta},1+u,1-u\ge 0$$

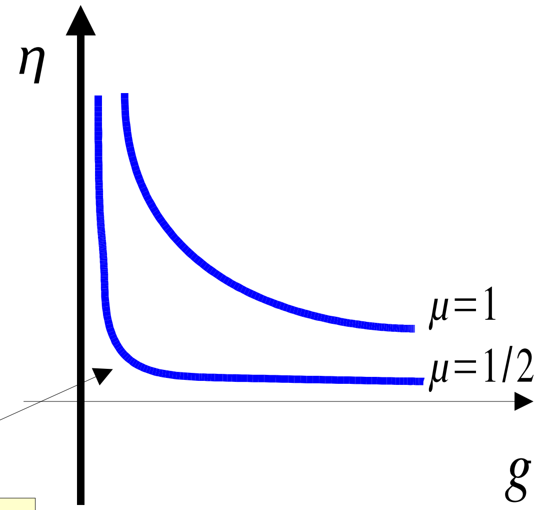


nondifferentiable

Primal-Dual Interior Point Method

$$
\begin{aligned}
y - y_d + L\lambda &= 0 \\
\alpha u + \lambda - \underline{\eta} + \overline{\eta} &= 0 \\
Ly + u &= 0 \\
\underline{\eta}(1+u) &= \mu \\
\overline{\eta}(1-u) &= \mu \\
\underline{\eta}, \overline{\eta}, 1+u, 1-u &\geq 0
\end{aligned}
$$

differentiable for $\mu > 0$

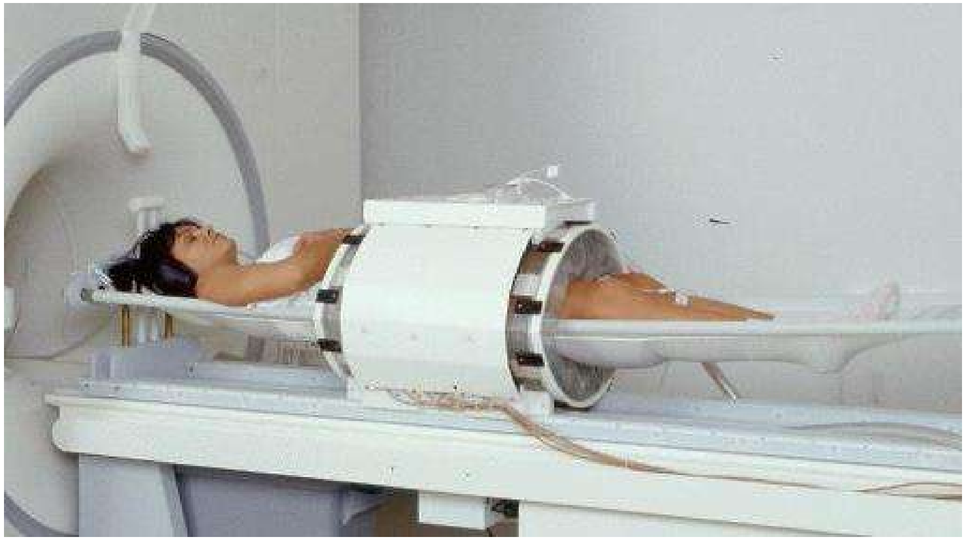


$v = (y, u, \lambda, \underline{\eta}, \overline{\eta})$

Homotopy in $\mu \to 0$ defines the central path $v(\mu)$

$F(v(\mu); \mu) = 0$

$\|v(\mu) - v(0)\|_{L_\infty} \leq \text{const} \sqrt{\mu}$



Medicine

- cancer therapy
- tumor heating by microwaves
- temperature constraints for healthy tissue

Mathematical Modelling

- time harmonic Maxwell equations
- nonlinear Bio-Heat-Transfer-Equation
- control and state constraints

Optimization

- optimization of antenna parameters
- identification of perfusion

$$\min \int_{\text{tumor}} (T - T_t)^2 \, d\xi$$

subject to

$$-\operatorname{div}(\kappa \nabla T) + c\, W(T)(T - T_a) = \mathrm{SAR}(u)$$

$$T \leq T_{\lim}(\xi)$$

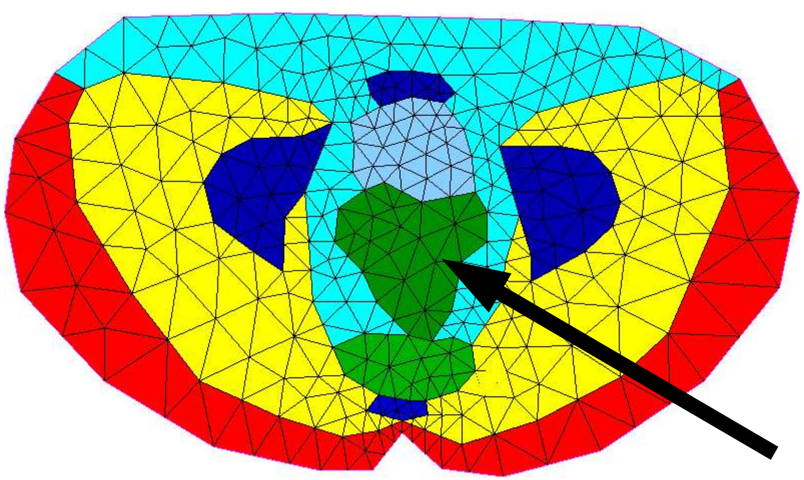
$$a \leq u \leq b$$

$$\min \int_{\Omega} (T - T_t)^2 + \alpha u^2 \, d\xi$$

subject to

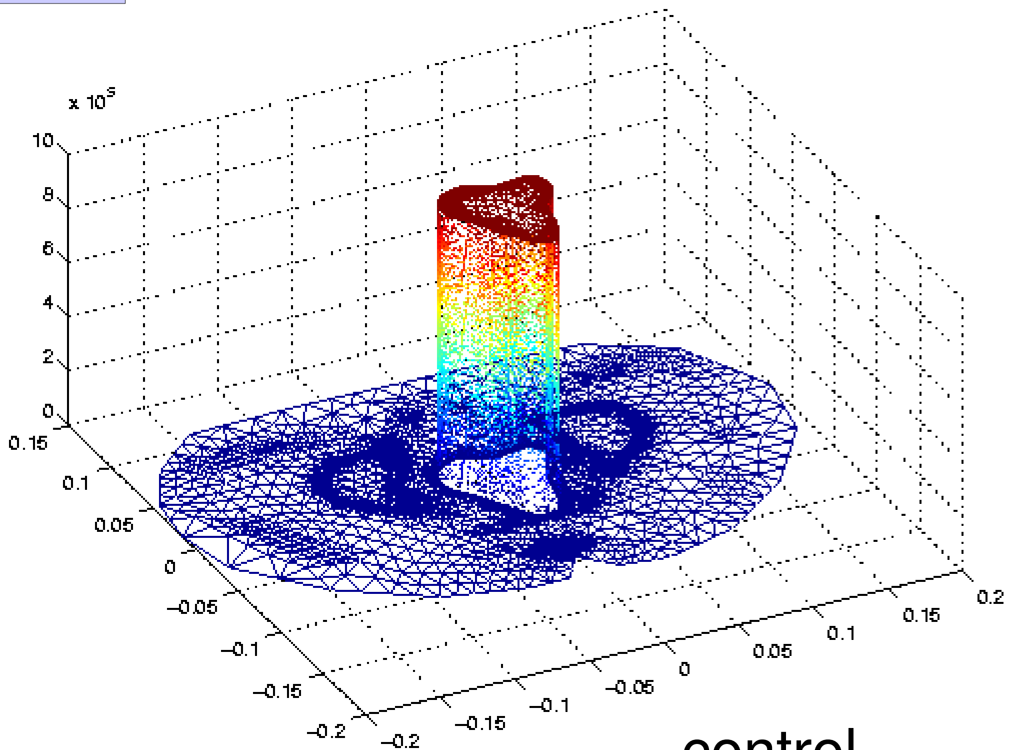
$$-\mathrm{div}(\kappa \nabla T) + c\,W(T)(T - T_a) = \mathrm{SAR}(u)$$

$$a \le u \le b$$

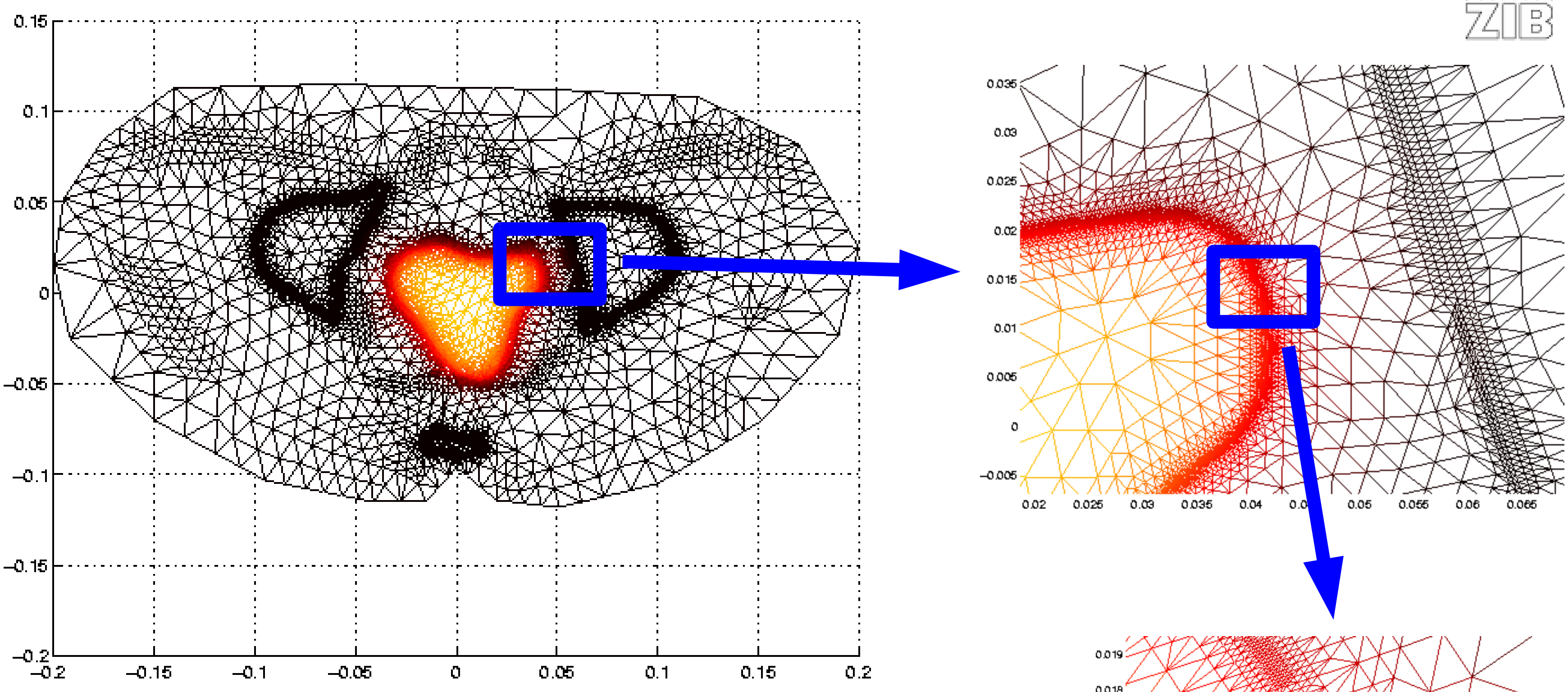


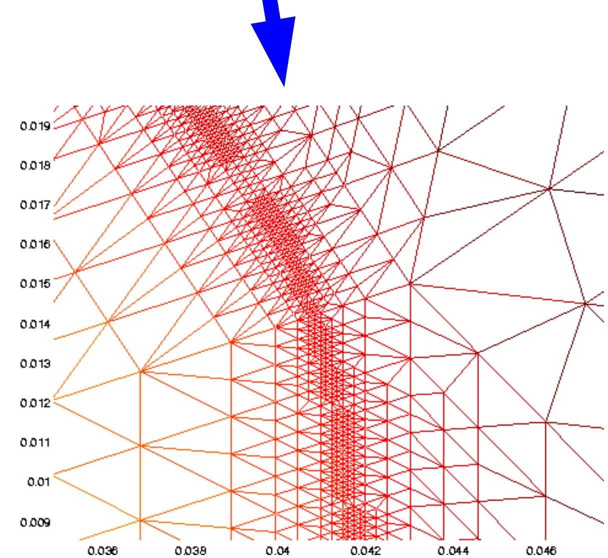
tumor

cross section of pelvic region

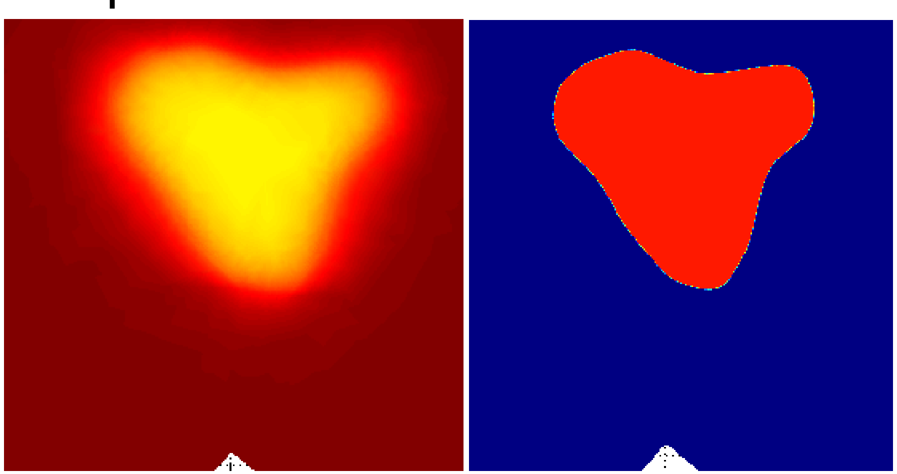


control





temperature control



$$\min_{u \in L^\infty,\, y \in H_0^1 \cap L^\infty} \frac{1}{2}\|y - y_d\|_{L_2}^2 + \frac{\alpha}{2}\|u\|_{L_2}^2$$

subject to $Ly + u = 0$

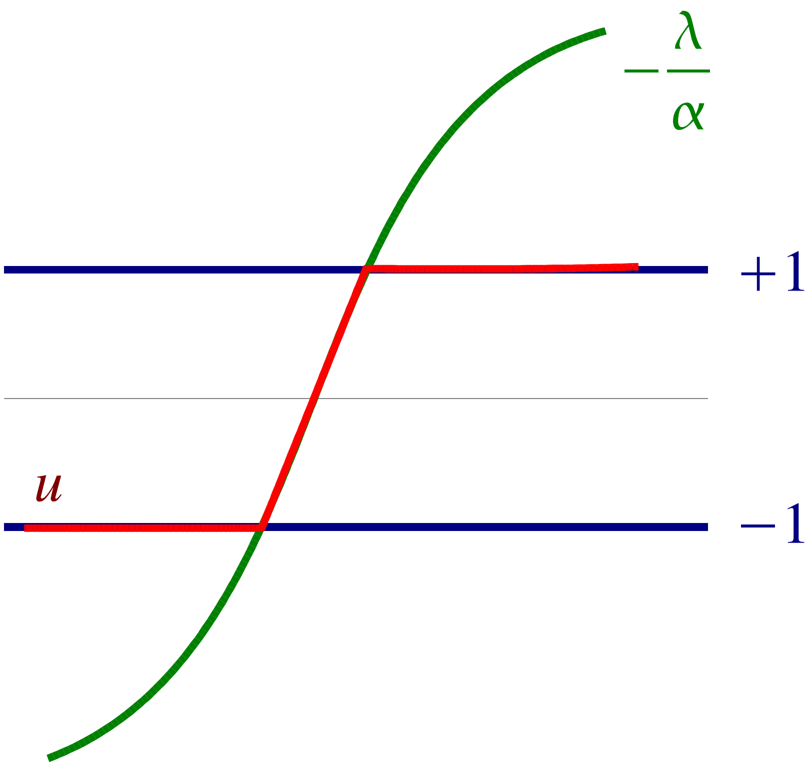
$$-1 \le u \le 1$$

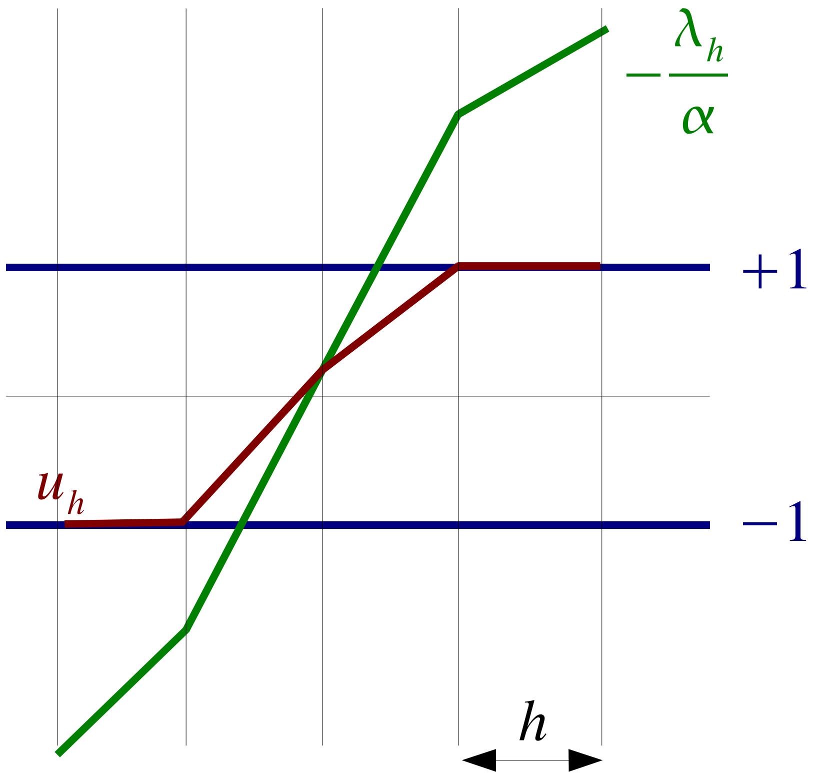
Optimality condtions:

$$y - y_d + Ly = 0$$

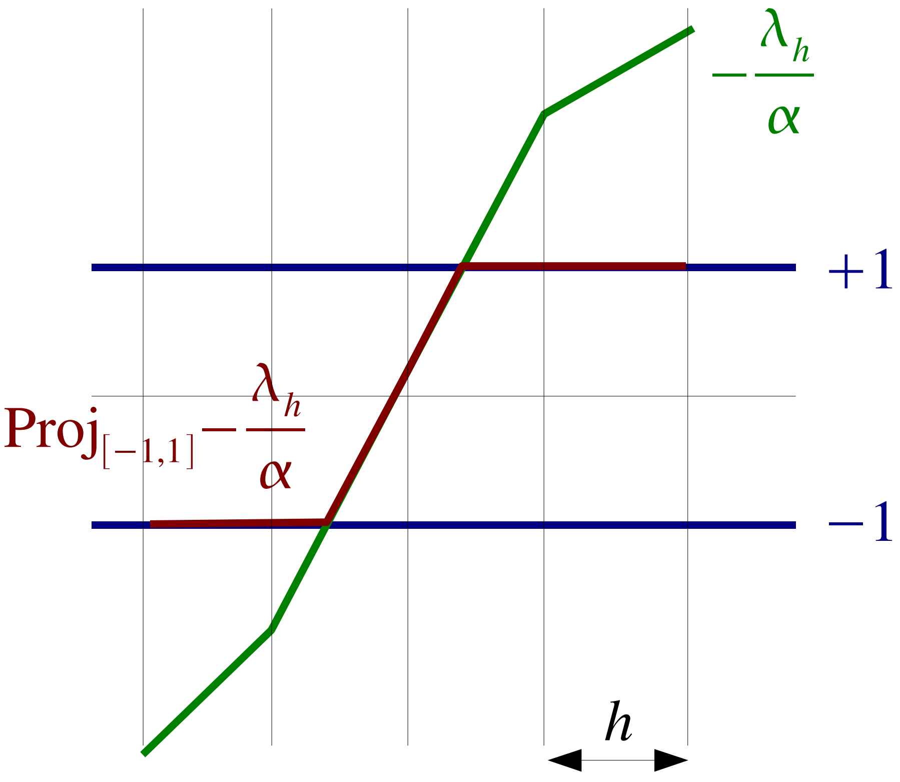
$$u = \mathrm{Proj}_{[-1,1]} - \frac{\lambda}{\alpha}$$

$$Ly + u = 0$$





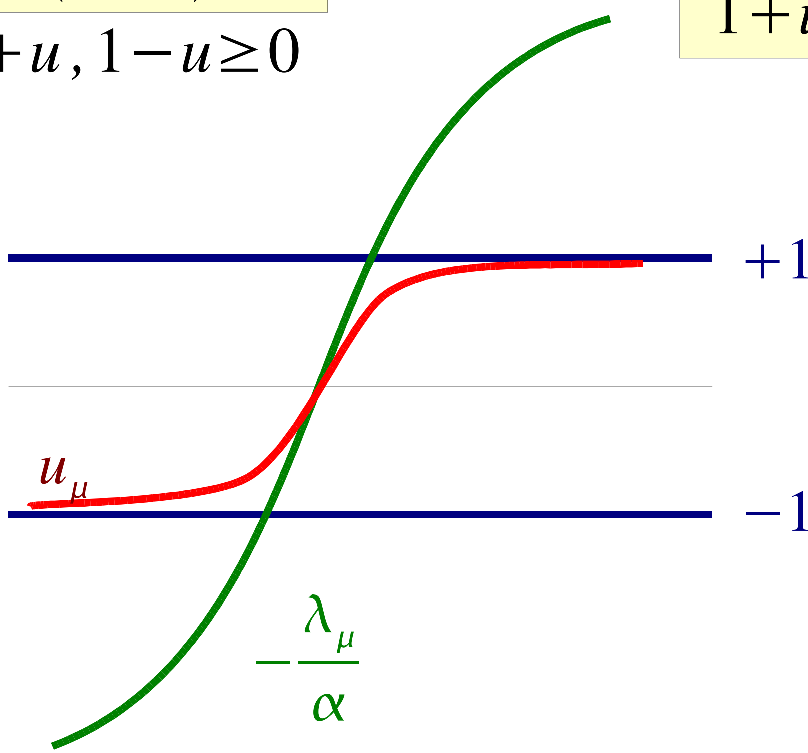
standard discretization



semi-discretization

Hinze: semismooth system
Rösch: postprocessing

Primal-Dual IP	Primal IP	Control Reduced Primal IP
$y-y_d+L\lambda=0$	$y-y_d+L\lambda=0$	$y-y_d+L\lambda=0$
$\alpha u+\lambda-\underline{\eta}+\overline{\eta}=0$	$\alpha u+\lambda-\frac{\mu}{1-u}+\frac{\mu}{1+u}=0$	$Ly+u(\lambda;\mu)=0$
$Ly+u=0$	$Ly+u=0$	
$\underline{\eta}(1+u)=\mu$	$1+u,1-u\geq 0$	
$\overline{\eta}(1-u)=\mu$		
$\underline{\eta},\overline{\eta},1+u,1-u\geq 0$		



$u(\lambda;\mu)$: unique solution of cubic equation can be evaluated pointwise triangulation of Ω: $\mathbf{T}$ $\partial\Omega$ smooth

piecewise polynomial functions: $\boldsymbol{P}_{p,h} = \{\phi \in L^2 : \phi|_T \in \boldsymbol{P}_p \; \forall T \in \mathbf{T}\}$

ansatz space: $V_h^p = \boldsymbol{P}_{p,h} \cap H_0^1$ $\quad p = 1, 2$

weak formulation:

$$\langle y_h - y_d + L\lambda_h, \phi\rangle = 0$$
$$\langle L y_h - P_h u(\lambda_h), \phi\rangle = 0 \qquad \forall \phi \in V_h^p$$

exact integration

numerical integration: linear projector P_h

$$\|P_h u - u\|_{L^2} \le c\, h^p \sqrt{\mu} \|u\|_{H^{2,}\mathbf{T}}$$

Result:

$$\|(y,\lambda)_h - (y,\lambda)(\mu)\|_{H^1} \le ch^p \qquad \|(y,\lambda)_h - (y,\lambda)(0)\|_{H^1} \le ch^p$$
$$\|u_h - u(\mu)\|_{L^2} \le ch^{p+1} \qquad \|u_h - u(0)\|_{L^2} \le ch^p$$

$$\min \frac{1}{2}\|y-y_d\|^2+\frac{\alpha}{2}\|u\|^2$$

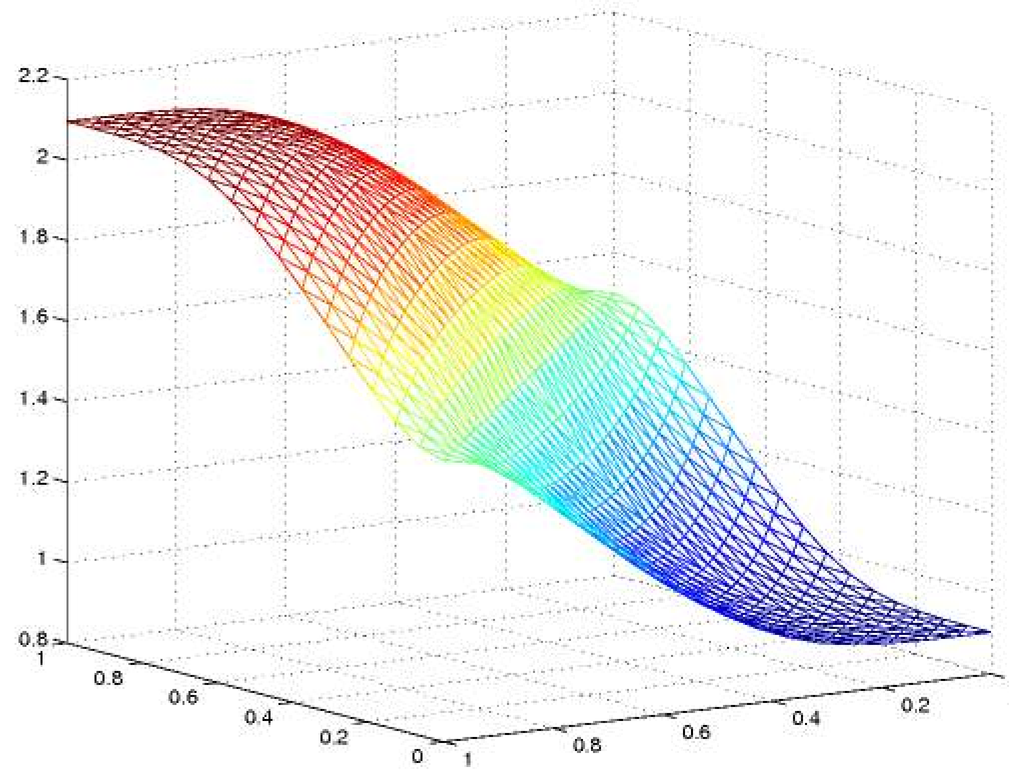
$$-\Delta y = u \qquad \Omega$$

$$\partial_n y = 0 \qquad \partial\Omega$$

$$|u| \le 6 \qquad \Omega = [0,1]^2$$

$$y_d = \begin{cases} 1, & x_1 + x_2 < 1 \\ 2, & \text{otherwise} \end{cases}$$

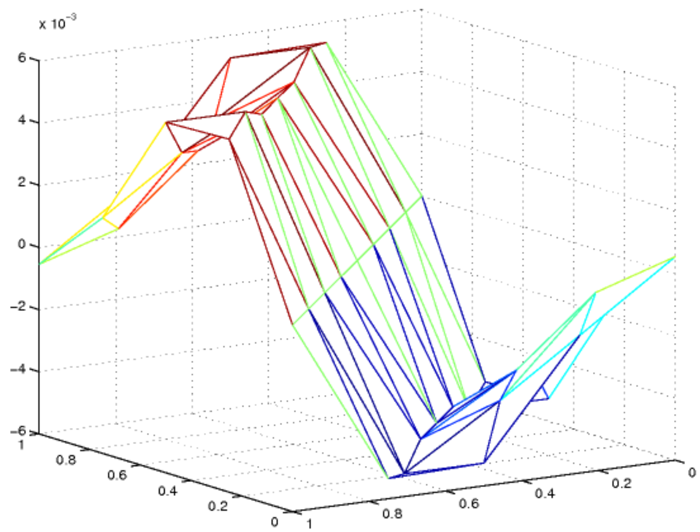
$$\alpha = 5 \cdot 10^{-4}$$

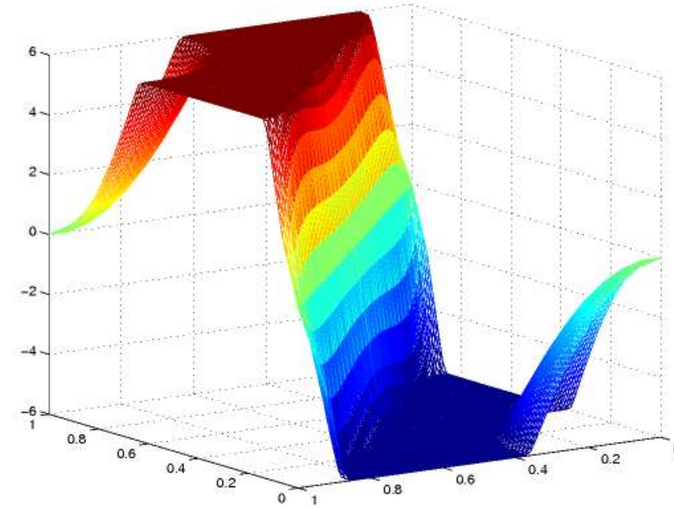


state at $\mu = 2^{-18}$

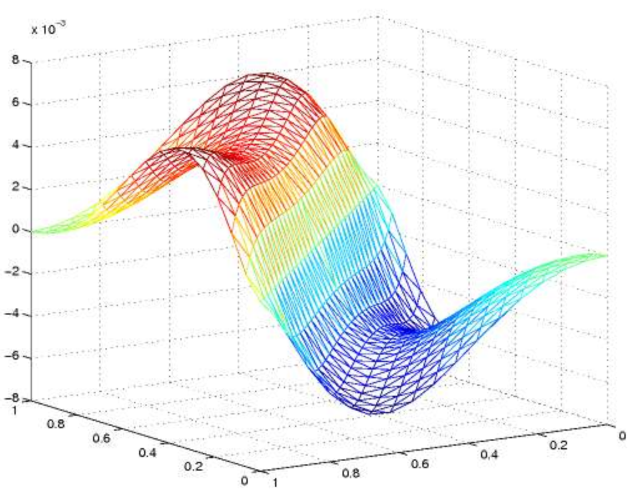
Illustrative Example

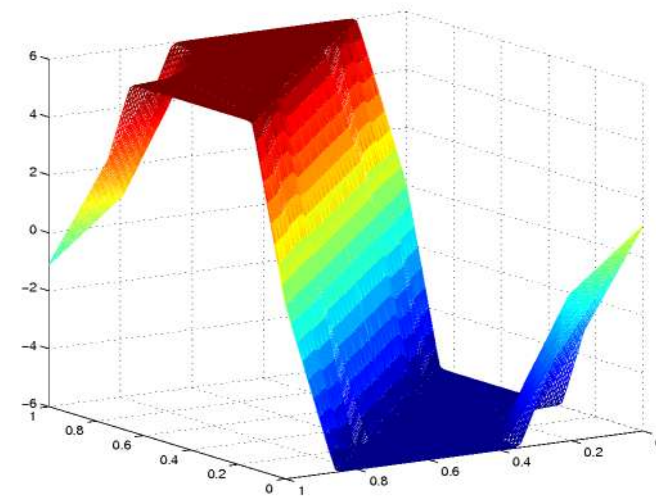
$$h=\frac{1}{4}$$



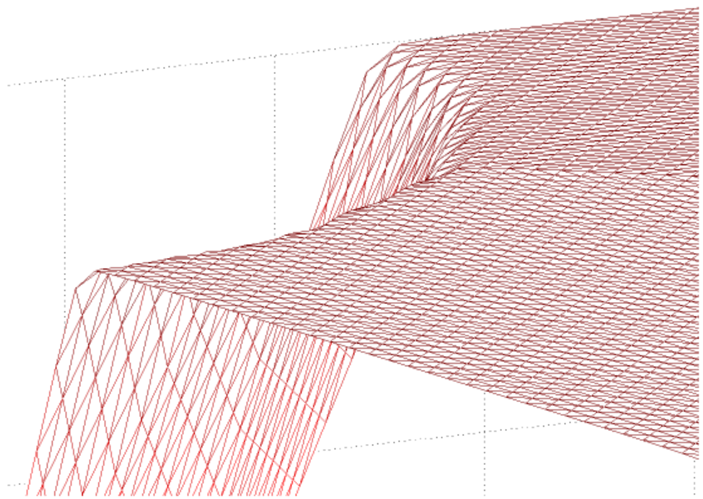


$$h=\frac{1}{16}$$

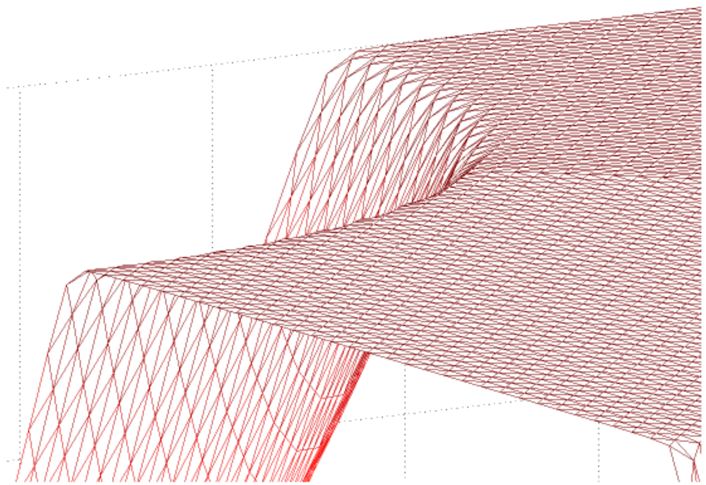




Zoom of control



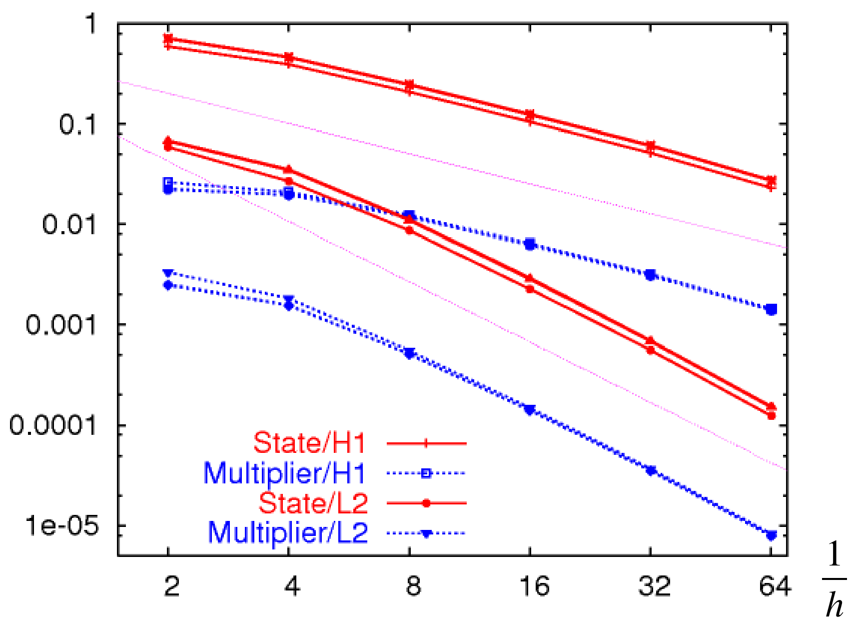
$$h=\frac{1}{4}$$



$$h=\frac{1}{16}$$

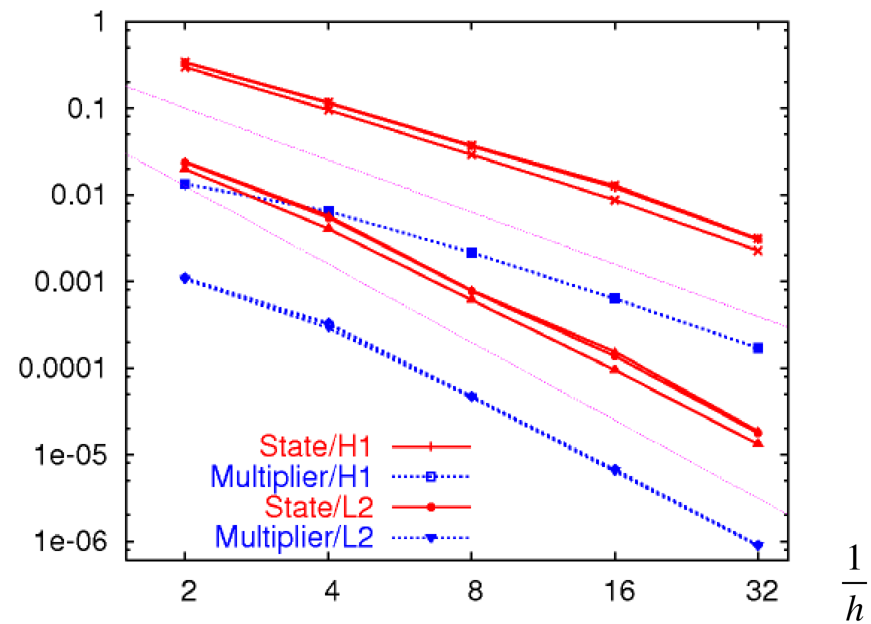
Estimated discretization error

$$\|y_h - y(\mu)\|$$
$$\|\lambda_h - \lambda(\mu)\|$$



linear finite elements

$p=1$

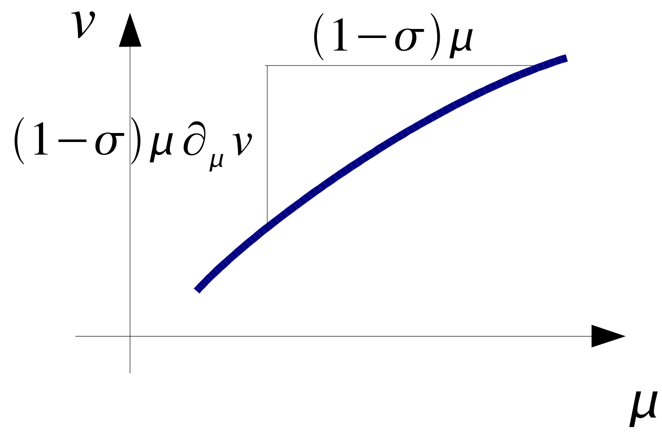


quadratic finite elements

$p=2$

Short step pathfollowing $\mu_{k+1} = \sigma(\mu_k)\mu_k$

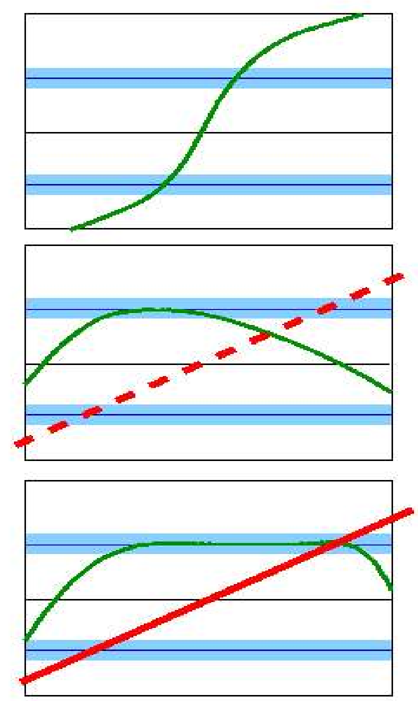
corrector convergence: $(1-\sigma)\mu\,\partial_\mu v \le \frac{1}{\omega}$



Generic:

Slope of central path $\partial_\mu v = O(\mu^{-1/2})$
Lipschitz constant $\omega \le O(\mu^{-1/2})$

$$\sigma(\mu) = c < 1$$



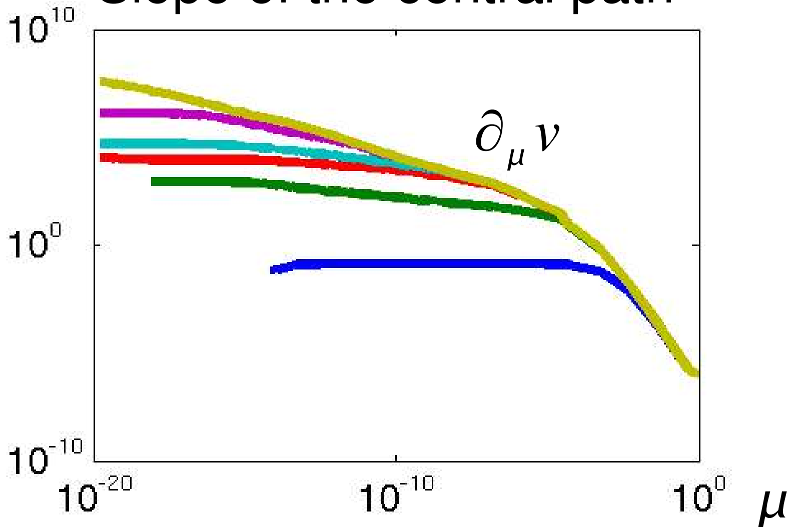
With strong strict complementarity:

$$\left|\left\{x \in \Omega : |\lambda(x) \pm \alpha| \le e\right\}\right| \le \Gamma e$$

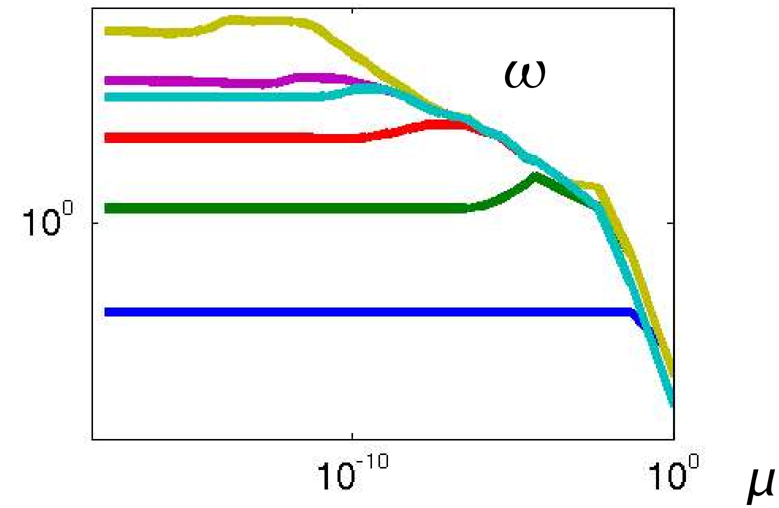
Slope of central path $\partial_\mu v = O(-\ln\mu)$
Lipschitz constant $\omega \le O(1)$

$$\sigma(\mu) = O(-\mu^2 \ln\mu)$$

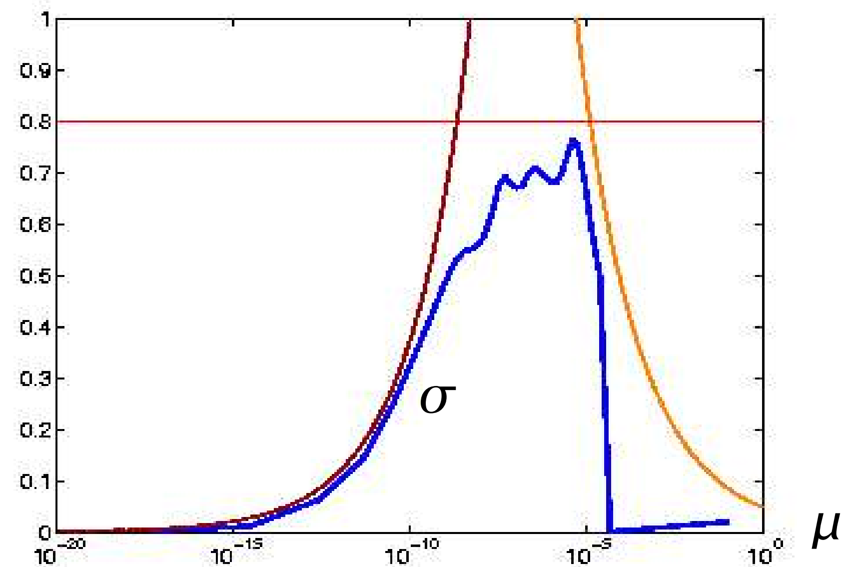
Slope of the central path



Lipschitz constant



$$\sigma \sim (\mu\,\omega\,\partial_\mu v)^{-1}$$



... the collaborators:

- Anton Schiela, Tobias Gänzler, Peter Deuflhard (ZIB & MATHEON)
- Fredi Tröltzsch, Uwe Prüfert (TUB & MATHEON)
- Peter Wust, Johanna Gellermann (Charité Berlin)

... you for your attention!