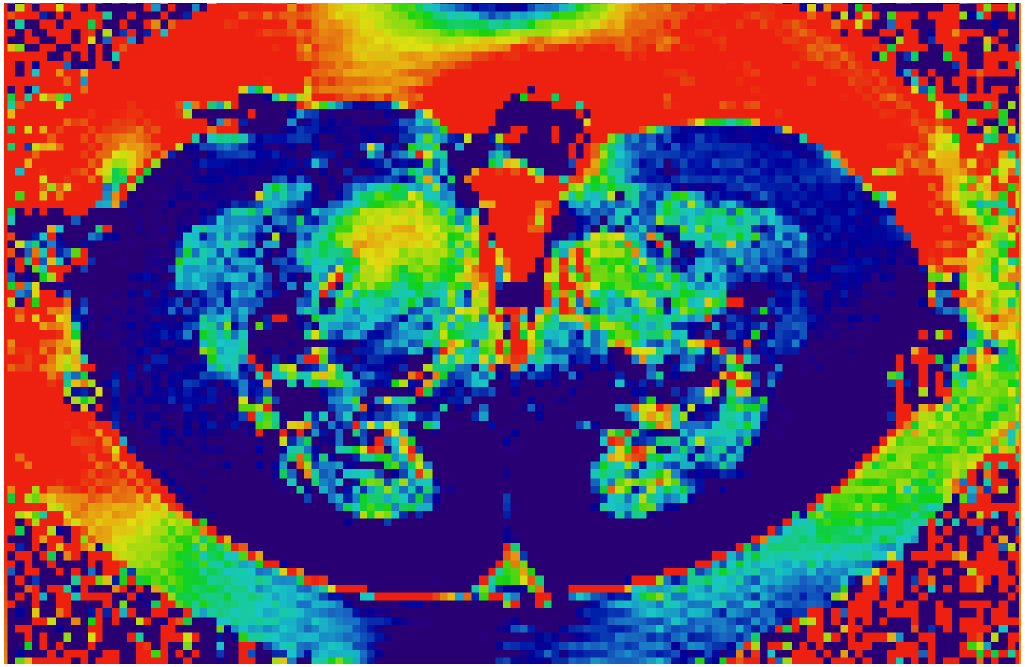


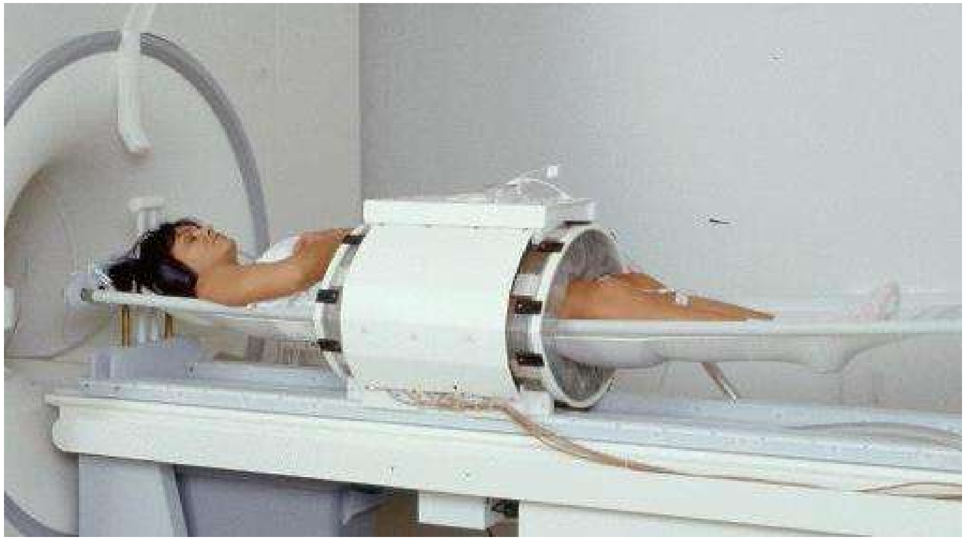
Martin Weiser

Optimization and Identification in Hyperthermia Treatment Planning

Aspects in Interior Point Methods for Optimal Control

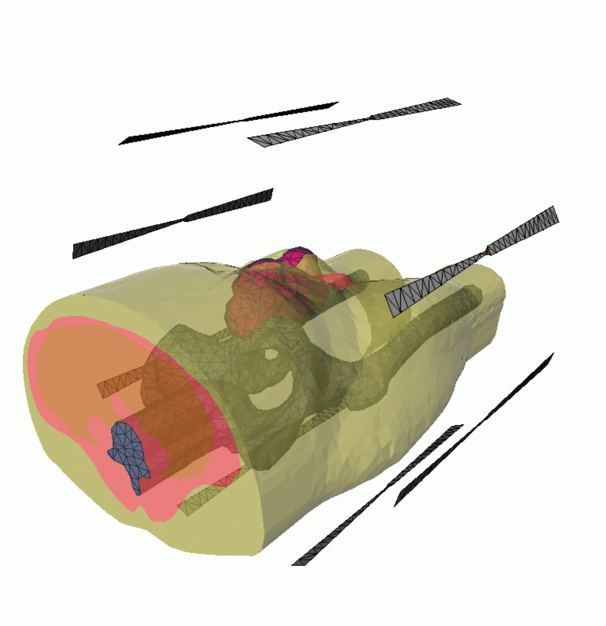
Principle

- tumors are susceptible to heat
- support radio- or chemotherapy by heating tumors



Technology for Regional Hyperthermia

- phased array microwave radiation
- ultrasound
- magnetic nanoparticle fluids



Mathematical Modelling

- virtual patient models
- Maxwell's equations / elastoacoustics
- bio-heat-transfer equation

Optimization Task

Find a control $u \in \mathbb{C}^{12}$ that optimizes the therapy.

cost functional

$$\min \quad J(T|_{\text{tumor}})$$

state equations

$$-\operatorname{div}(\kappa \nabla T) + c\, W(T)(T - T_a) = \frac{\sigma}{2}|E|^2$$

$$\kappa \partial_n T = h(T_{\text{out}} - T)$$

BHTE

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} E = \omega^2 \epsilon E - i \omega J(u)$$

time-harmonic Maxwell's equations

constraints

$$T|_{\text{healthy}} \le T_{\max}$$

$$\min\{|u_i|\} \ge \alpha \max\{|u_i|\}$$

- fixed control setting: finite dimensional control $u \in \mathbb{C}^{12}$
- periodic control setting: distributed control $u \in L_\infty^{12}(0, \pi)$

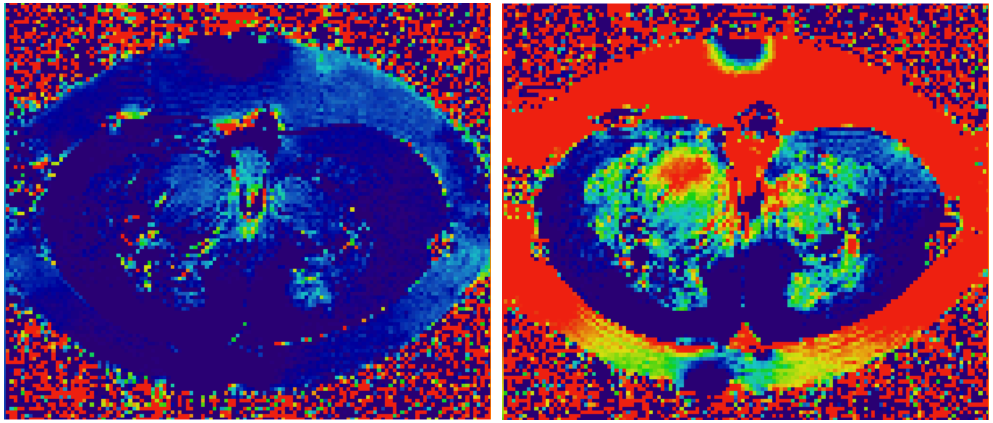
Sources of planning error

- individual and varying perfusion
- patient positioning
- generator and transmitter drift
- modelling errors
- material parameters
- numerical approximation errors

Is the therapy plan realized in practice?

online MR "temperature" measurements

$$m = a(T - T_{\text{bas}}) + b(W - W_{\text{bas}}) + \delta$$



- noisy
- only partially available
- coefficients a and b not exactly known

Parameter identification

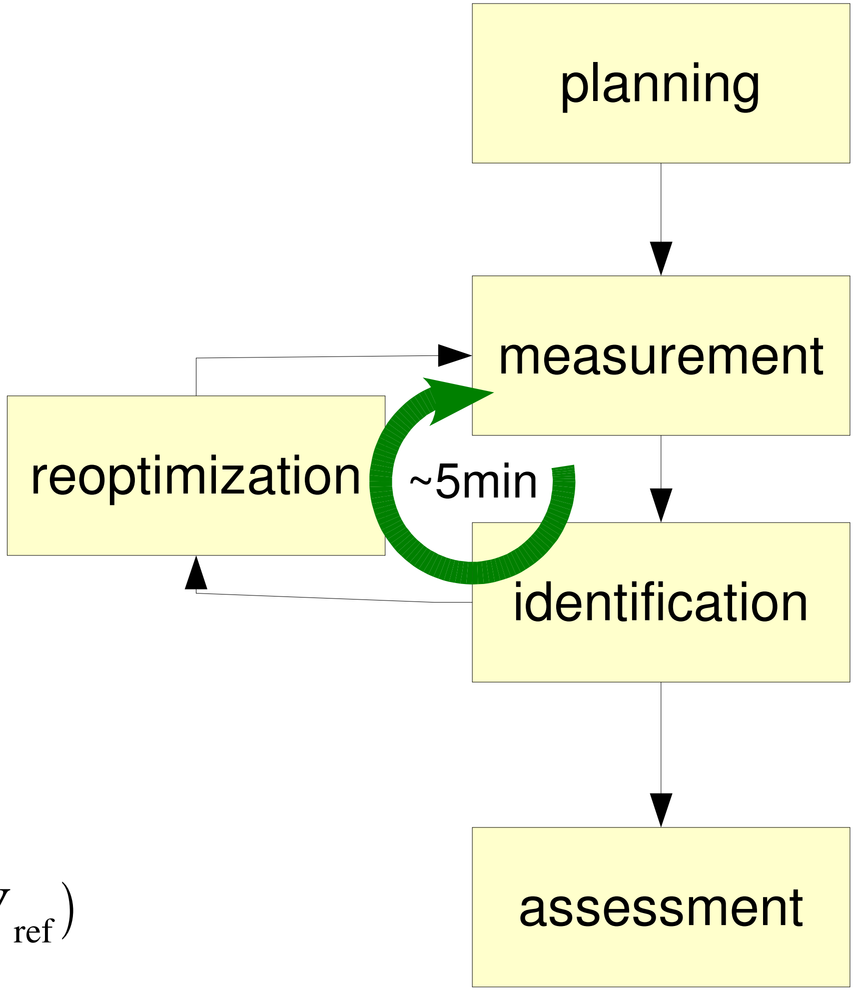
$$\min_{T,W} \; \frac{1}{2}\|m - a(T - T_{\text{bas}}) - b(W - W_{\text{bas}})\|^2 + \frac{\alpha}{2}\|W - W_{\text{ref}}\|^2 + \frac{\beta}{2}\sum_i \|\nabla(W - W_{\text{ref}})\|^2_{\Omega_i}$$

$$\text{s.t.} \quad \text{BHTE}, \; W \geq 0$$

Sources of planning error

- individual and varying perfusion
- patient positioning
- generator and transmitter drift
- modelling errors
- material parameters
- numerical approximation errors

Online reoptimization of therapy



Parametric sensitivities

sensitivity $\partial_W u_{\text{opt}}$

update $u_{\text{opt}}(W) \approx u_{\text{ref}} + \partial_W u_{\text{opt}} (W - W_{\text{ref}})$

Model problem
$$\min_{u\in L^\infty,\, y\in H_0^1\cap L^\infty} \frac{1}{2}\|y-y_d\|_{L_2}^2+\frac{\alpha}{2}\|u\|_{L_2}^2$$

subject to $$L\,y+u=0$$

$$-1\le u\le 1$$

IP regularization
$$\min_{u\in L^\infty,\, y\in H_0^1\cap L^\infty} \frac{1}{2}\|y-y_d\|_{L_2}^2+\frac{\alpha}{2}\|u\|_{L_2}^2-\mu\int_\Omega \ln(1+u)+\ln(1-u)$$

subject to $$L\,y+u=0$$

Homotopy $\mu\to 0$

$$\|u(\mu)-u(0)\|_\infty\le\sqrt{\mu}$$

Classical (inexact) continuation

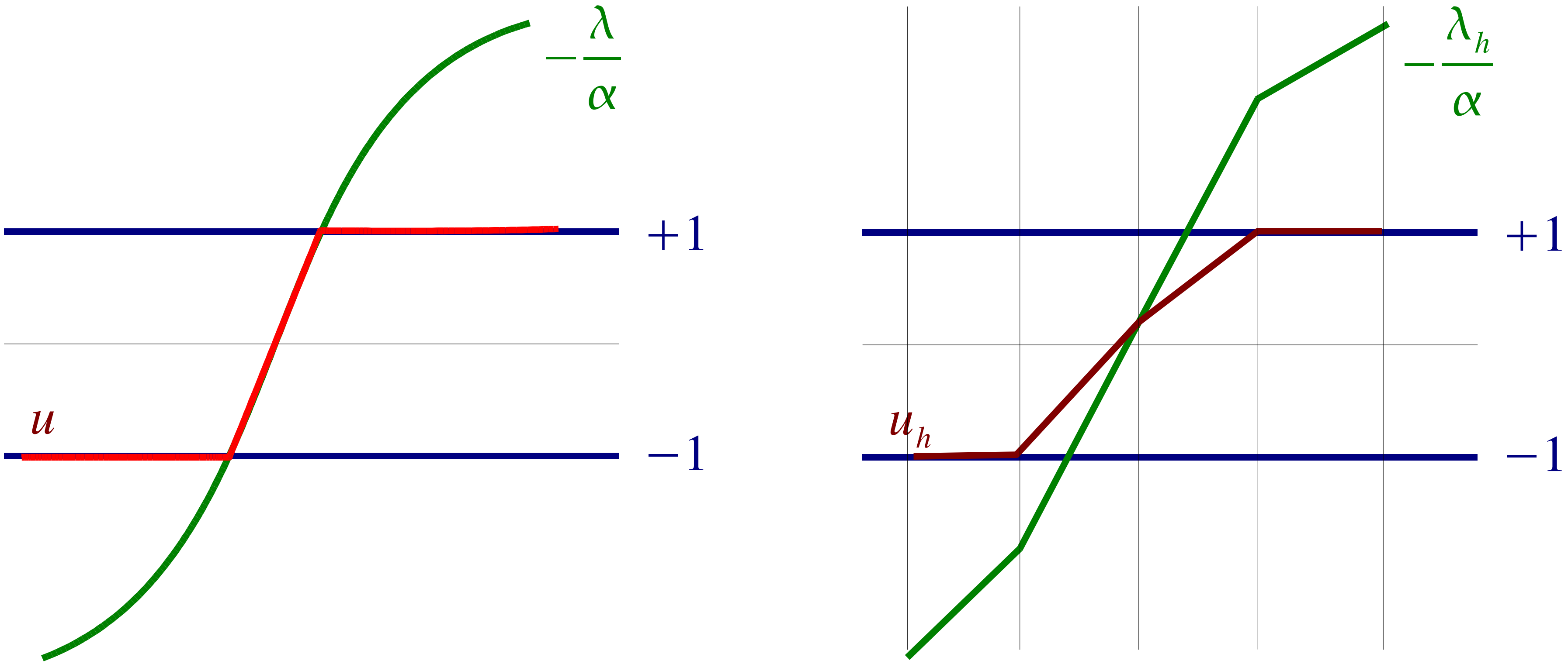
$\mu_{k+1}\le\sigma\,\mu_k$ linear convergence

Projection formula

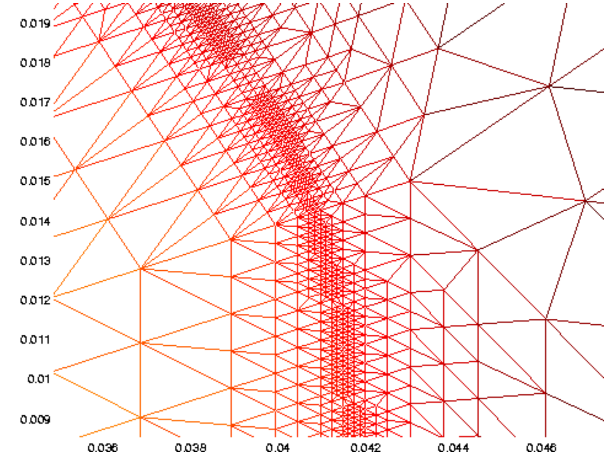
$$y - y_d + Ly = 0$$

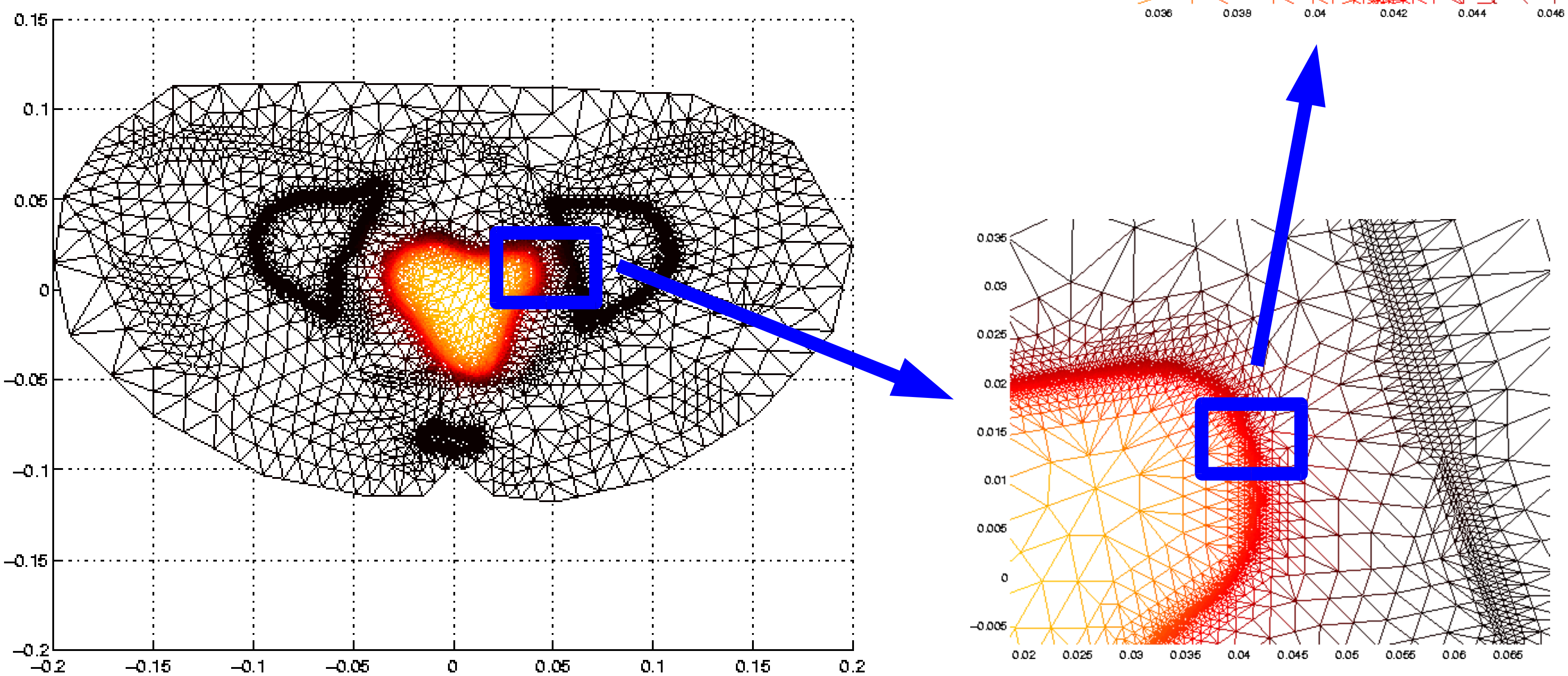
$$u = \mathrm{Proj}_{[-1,1]} -\frac{\lambda}{\alpha}$$

$$Ly + u = 0$$

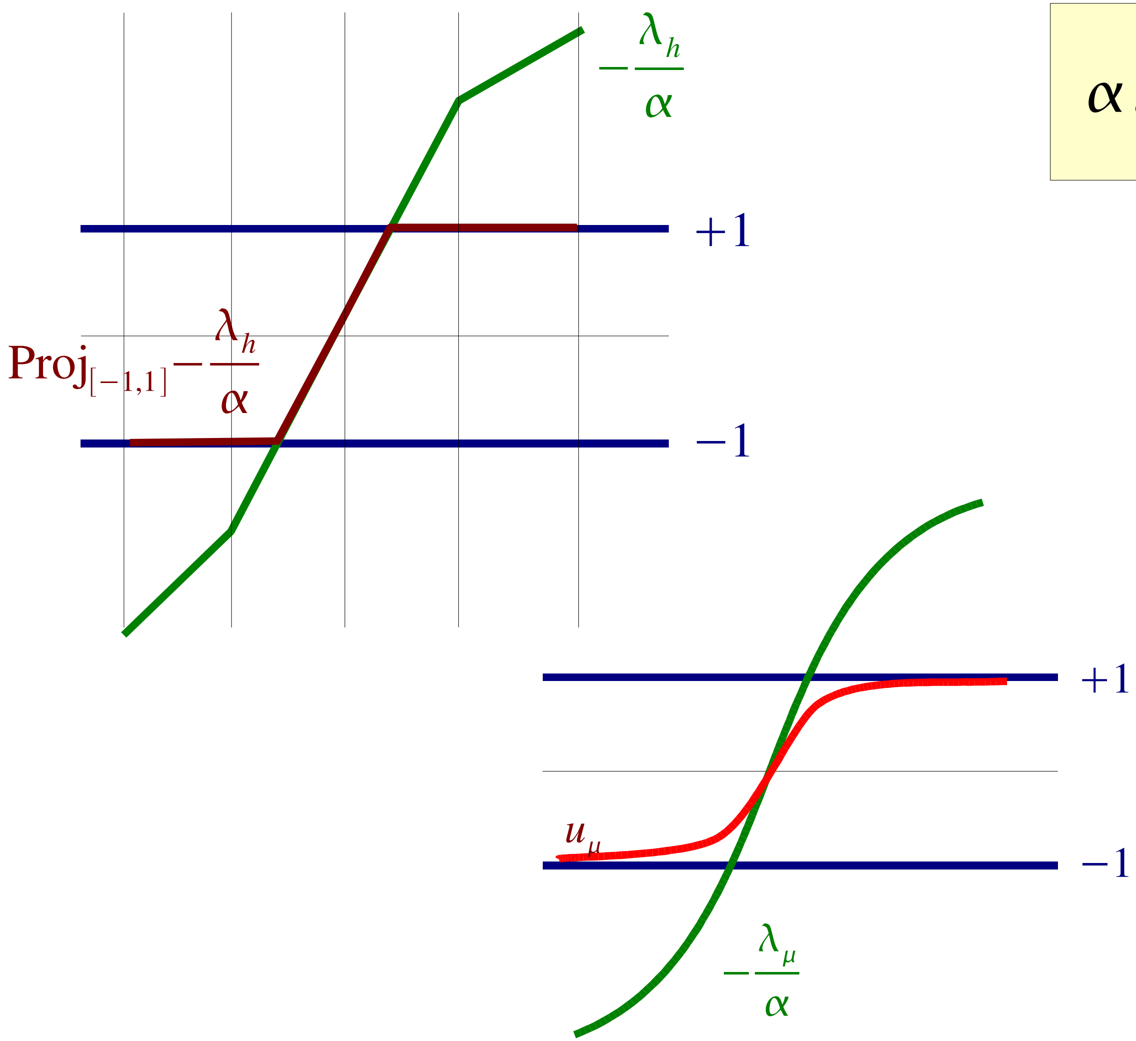


Accurate representation of control necessitates massive grid refinement at boundary of active set.





Pointwise elimination of control



IPM equivalent

$$y - y_d + L\lambda = 0$$

$$\alpha u + \lambda - \frac{\mu}{1-u} + \frac{\mu}{1+u} = 0$$

$$Ly + u = 0$$

$$1+u,\ 1-u \geq 0$$

$$\Downarrow$$

$$y - y_d + L\lambda = 0$$

$$Ly + u(\lambda;\mu) = 0$$

$u(\lambda;\mu)$ solution of cubic equation triangulation of Ω: $\mathbf{T}$ $\partial\Omega$ smooth

piecewise polynomial functions: $\boldsymbol{P}_{p,h} = \{\phi \in L^2 : \phi|_T \in \boldsymbol{P}_p \ \forall\, T \in \mathbf{T}\}$

ansatz space: $V_h^p = \boldsymbol{P}_{p,h} \cap H_0^1 \qquad p = 1, 2$

weak formulation:

$$\langle y_h - y_d + L\lambda_h, \phi \rangle = 0$$
$$\langle L y_h - P_h u(\lambda_h), \phi \rangle = 0 \qquad \forall\, \phi \in V_h^p$$

exact integration

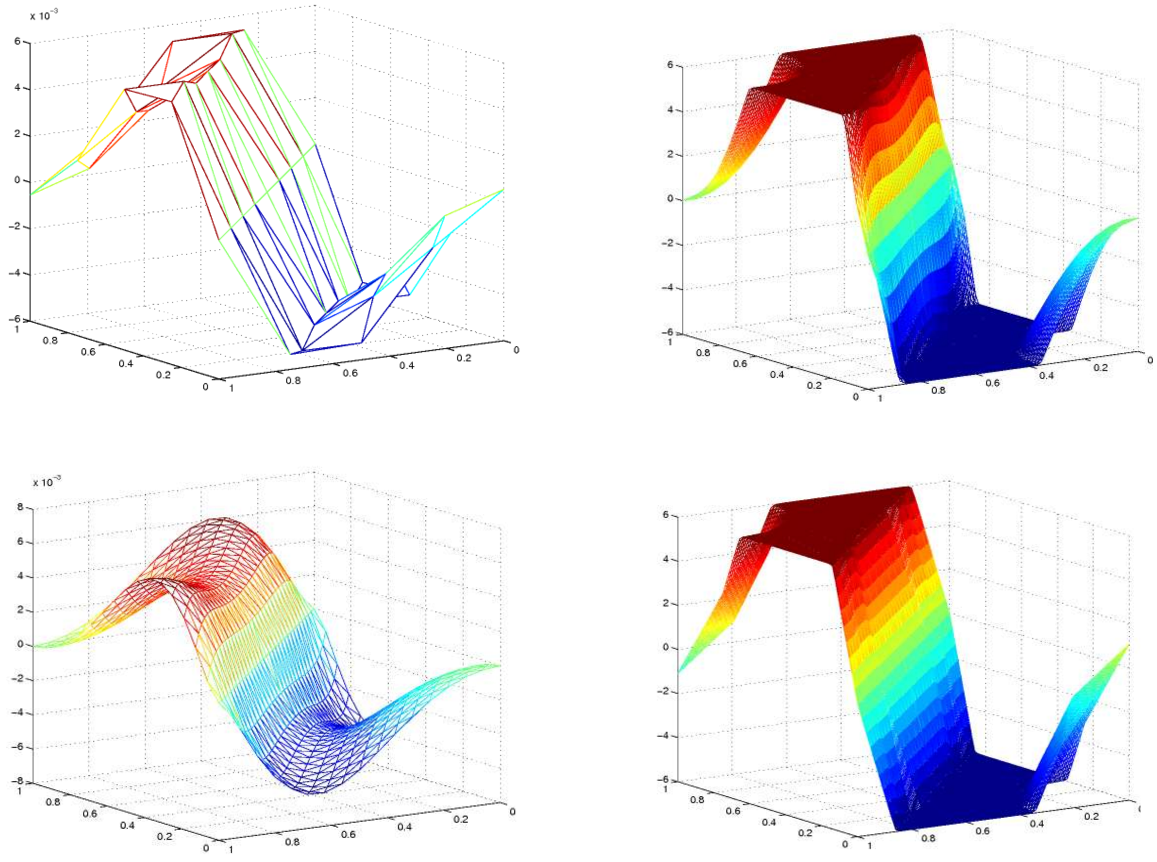
numerical integration: linear projector P_h

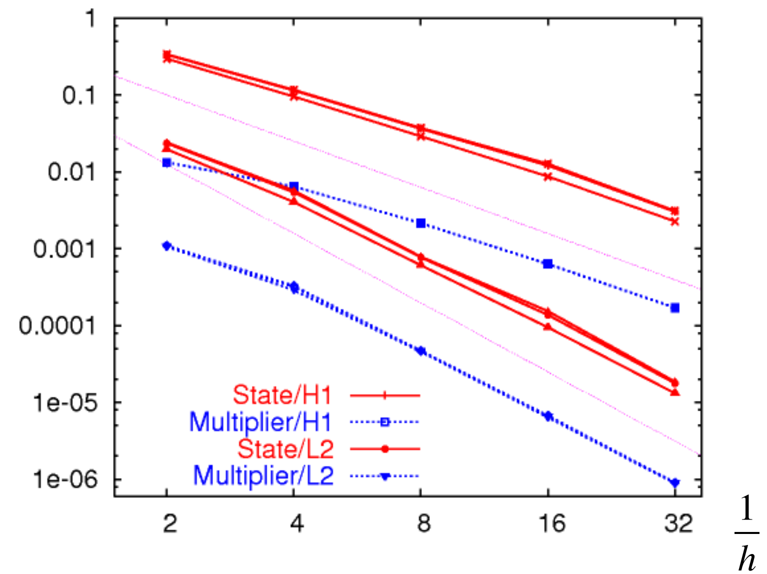
$$\|P_h u - u\|_{L^2} \le c\, h^p \sqrt{\mu}\, \|u\|_{H^{2},\mathbf{T}}$$

Result:

$$\|(y,\lambda)_h - (y,\lambda)(\mu)\|_{H^1} \le c h^p \qquad \|(y,\lambda)_h - (y,\lambda)(0)\|_{H^1} \le c h^p$$
$$\|u_h - u(\mu)\|_{L^2} \le c h^{p+1} \qquad \|u_h - u(0)\|_{L^2} \le c h^p$$

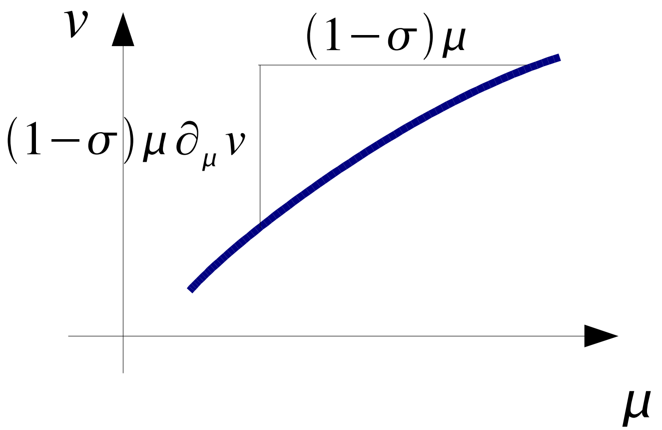
High accuracy on coarse grids





Superlinear convergence

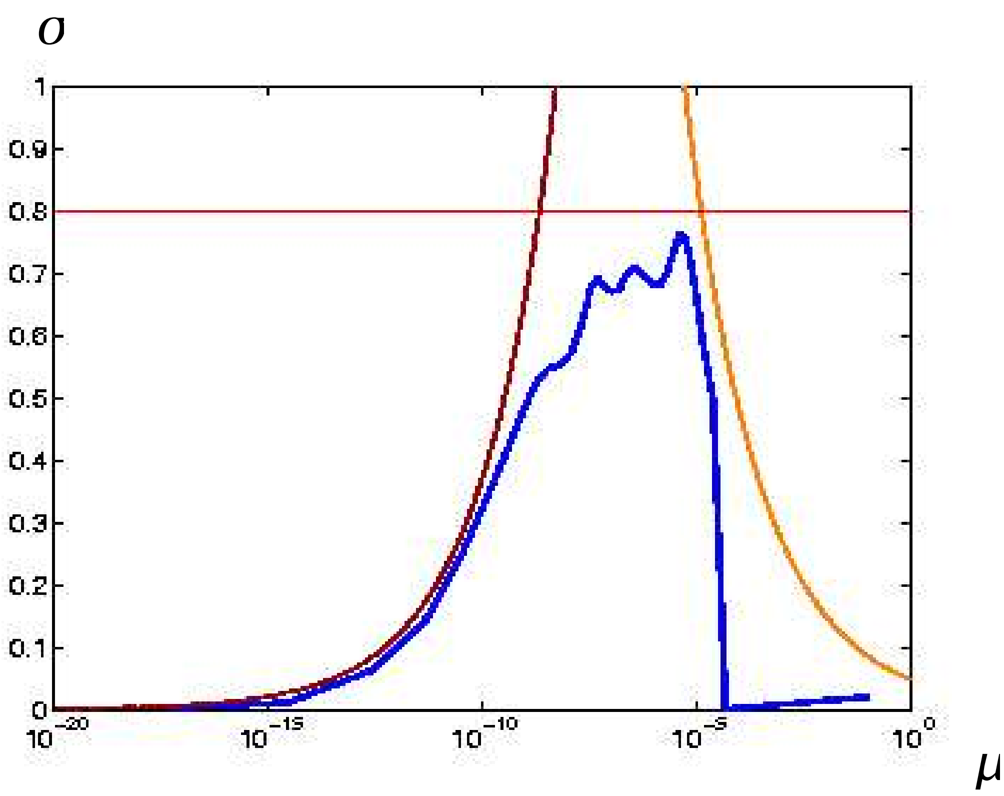
classical continuation



stepsize σ depends on

- slope of central path
- convergence radius of Newton corrector

$$\sigma(\mu) = O(-\mu^2 \ln \mu)$$



Control reduction not always possible

e.g. identification problem

$$J = \frac{1}{2}\|y - y_d\|^2 + \frac{\alpha}{2}\|u\|^2 + \frac{\beta}{2}\|\nabla u\|^2$$

$$Ly + u = 0, \quad u \geq 0$$

IP optimality system

$$\begin{aligned} y - y_d + L\lambda &= 0 \\ \alpha u - \Delta u + \lambda + \frac{\mu}{u} &= 0 \\ Ly + u &= 0 \\ u &\geq 0 \end{aligned}$$

Control discretization

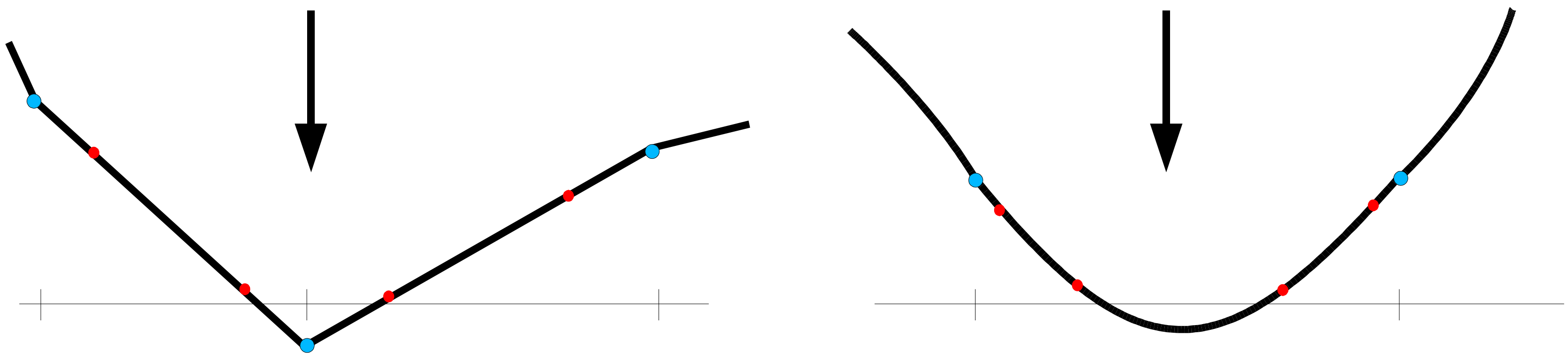
$$u_h \in \boldsymbol{P}_{p,h} = \{u : u|_T \in \boldsymbol{P}_p \; \forall\, T \in \boldsymbol{T}\}$$

Quadrature rules

assembly of barrier functional $\int_\Omega \ln u(x)\,dx = \sum_T \int_T \ln u(x)\,dx$

$$\int_T \ln u(x)\,dx \approx \sum_i w_i \ln u(x_i)$$

- discrete barrier functional: only $u(x_i) \geq 0$
- linear elements & linear constraints: use Lobatto quadrature $u(x) \geq 0\ \forall\, x$
- higher order elements:
  - perform feasibility check for Newton stepsize damping at integration points only or
  - perform mesh refinement



FE Discretization of Primal IPM

Exact integration

instead of $\int_T \ln u(x)\, dx \approx \int_T p(x)\, dx$ with $p(x_i) = \ln u(x_i)$

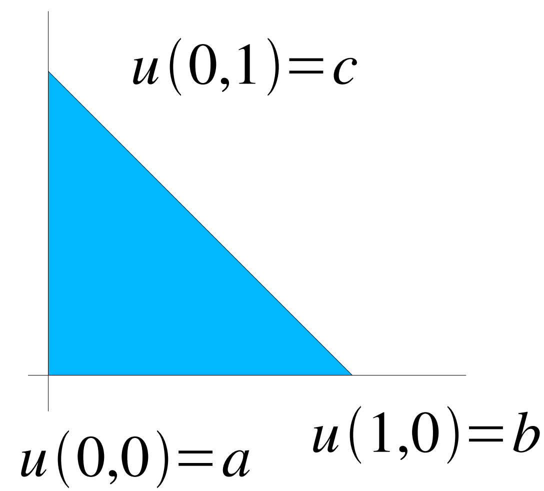
do $\int_T \ln u(x)\, dx \approx \int_T \ln q(x)\, dx$ with $q(x_i) = u(x_i)$

here: $u|_T \in P$ allows for exact integration

Example: linear 2D elements

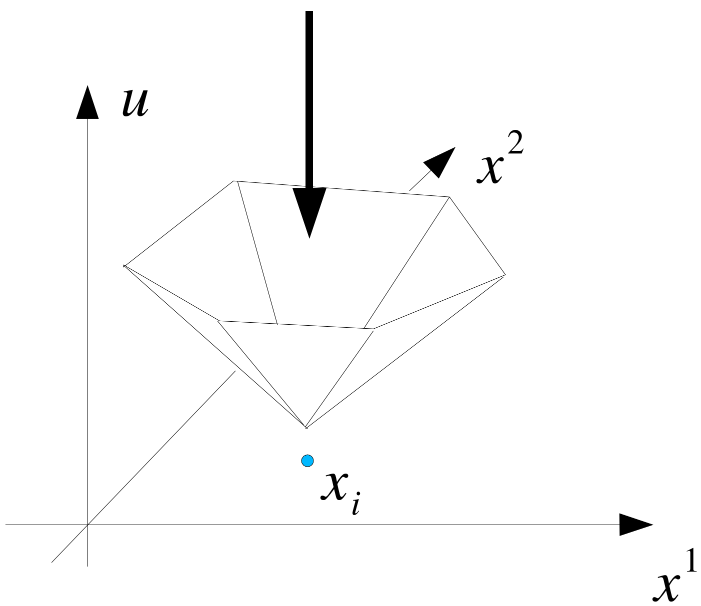
$$\int_T \ln u(x)\, dx = \frac{1}{c-a}\left(\frac{F(b)-F(c)}{b-c} + \frac{F(b)-F(a)}{b-a}\right)$$

$$F(z) = \frac{z^2}{2}\left(\ln z - \frac{3}{2}\right)$$



FE Discretization of Primal IPM

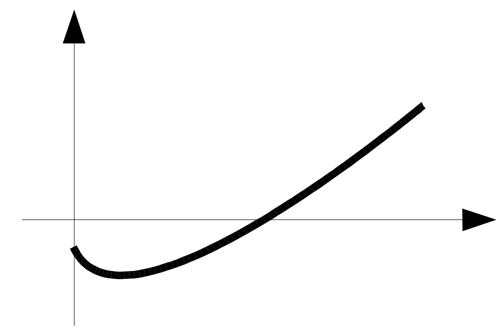
Newton damping with exact integration



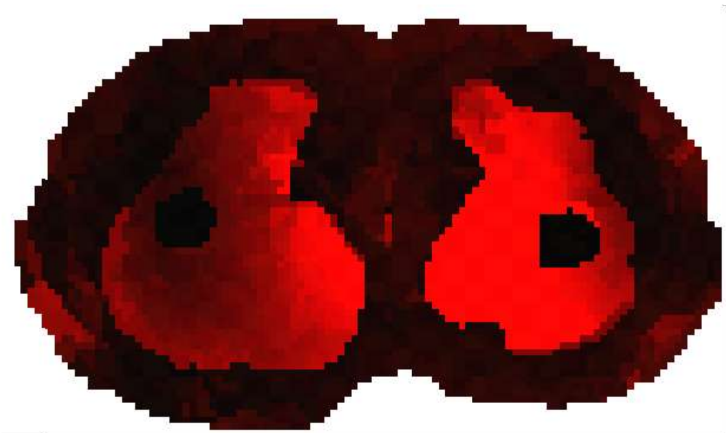
$$-\int_{T \ni x_i} \ln u(x)\, dx \approx -u(x_i) \ln u(x_i) =: f(u(x_i))$$

$$z \to 0: \quad f(z) \to \text{const}$$

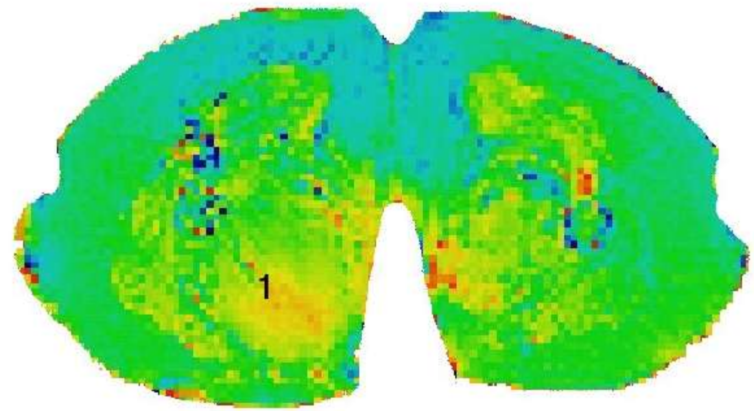
$$f'(z) \to -\text{const}$$



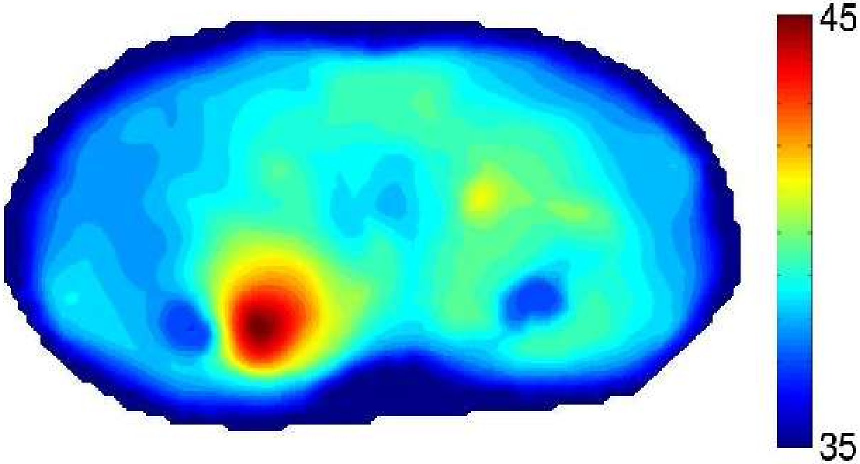
- discrete “barrier” functional need not have unconstrained minimizer
- mesh refinement necessary
- provides refinement criterion



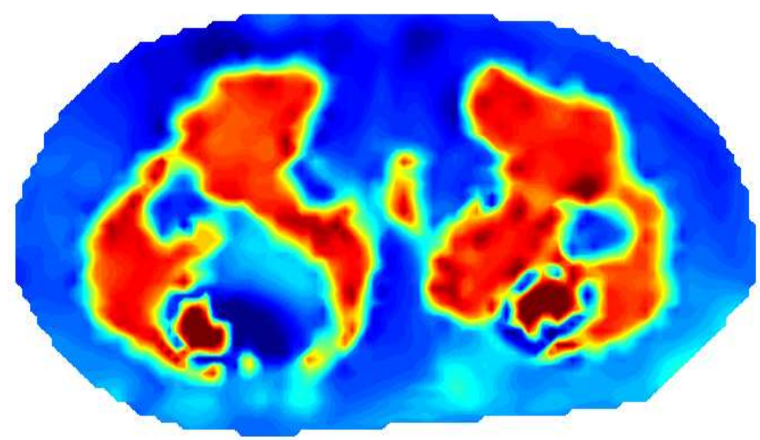
planning SAR



MR measurements



identified temperature



identified perfusion

Model Problem

$$\min \quad J(u;p) \quad \text{s.t. } u \geq \underline{u}$$

$$J(u;p) = \frac{1}{2}\langle u, K(p)u + l\rangle + \frac{\alpha}{2}\|u\|^2$$

Parametric Sensitivity

$$u(p+\delta p) = u(p) + u_p(p)\delta p + o(\delta p)$$

$$\min_{u_p \delta p} \langle u_p \delta p, J_{uu}(u;p) u_p \delta p + J_{up}\delta p\rangle$$

$$\text{s.t.} \quad u_p = 0, \quad \eta > 0$$

$$u_p \geq 0, \quad \eta = 0, u - \underline{u} = 0$$

IP Approximation

$$\min \quad J(u;p) - \mu \int_\Omega \ln(u - \underline{u})$$

gradient: $F(u;p,\mu)$

IP Sensitivity

$$u(p+\delta p, \mu) = u(p,\mu) + u_p(p,\mu)\delta p + o(\delta p)$$

$$u_p(p,\mu) = -F_u(u(p,\mu);p,\mu)^{-1} F_p(u(p,\mu);p,\mu)$$

Convergence

$$\min \int_{-1}^{1} (u-x-p)^2 \quad \text{s.t.} \quad u \geq 0$$

solution

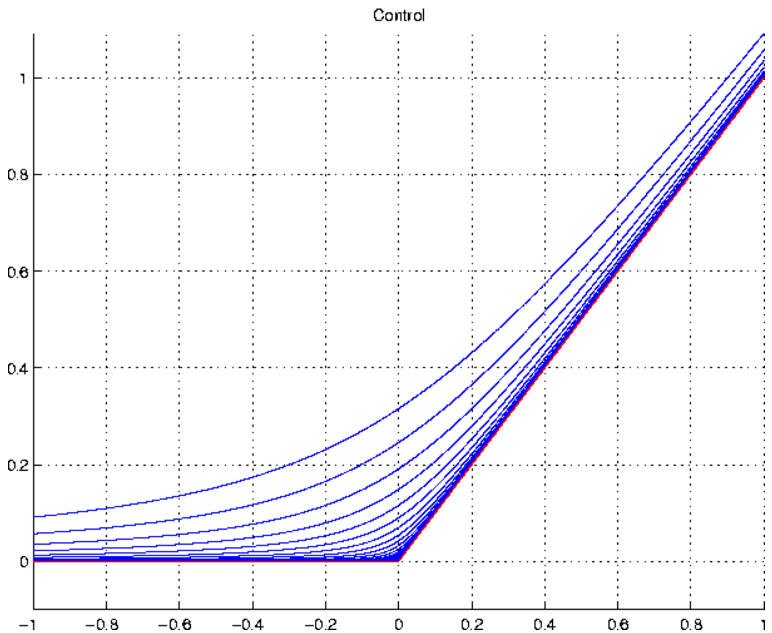
$$\|u(p,\mu)-u(p)\|_{L_q} \leq c\,\mu^{(1+q)/2q}$$

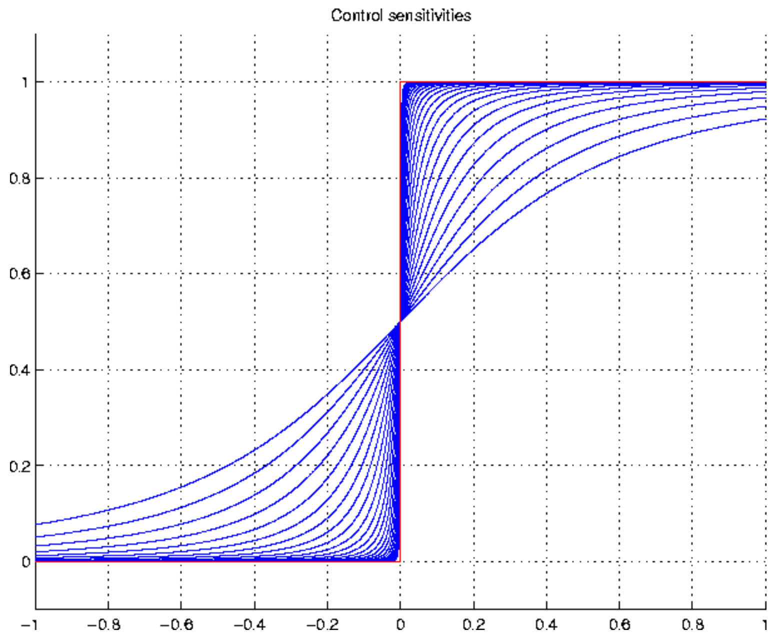
$$q=2 : \mu^{3/4}$$
$$q=\infty : \mu^{1/2}$$

sensitivity

$$\|u_p(p,\mu)-u_p(p)\|_{L_q} \leq c\,\mu^{1/2q}$$

$$q=2 : \mu^{1/4}$$
$$q=\infty : 1$$





Thanks to...

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